

TOPOLOGICAL PROPERTIES OF HAMILTONIAN CIRCLE ACTIONS

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ABSTRACT. This paper studies Hamiltonian circle actions, i.e. circle subgroups of the group $\text{Ham}(M, \omega)$ of Hamiltonian symplectomorphisms of a closed symplectic manifold (M, ω) . Our main tool is the Seidel representation of $\pi_1(\text{Ham}(M, \omega))$ in the units of the quantum homology ring. We show that if the weights of the action at the points at which the moment map is a maximum are sufficiently small then the circle represents a nonzero element of $\pi_1(\text{Ham}(M, \omega))$. Further, if the isotropy has order at most two and the circle contracts in $\text{Ham}(M, \omega)$ then the homology of M is invariant under an involution. For example, the image of the normalized moment map is a symmetric interval $[-a, a]$. If the action is semifree (i.e. the isotropy weights are 0 or ± 1) then we calculate the leading order term in the Seidel representation, an important technical tool in understanding the quantum cohomology of manifolds that admit semifree Hamiltonian circle actions. If the manifold is toric, we use our results about this representation to describe the basic multiplicative structure of the quantum cohomology ring of an arbitrary toric manifold. There are two important technical ingredients; one relates the equivariant cohomology of M to the Morse flow of the moment map, and the other is a version of the localization principle for calculating Gromov–Witten invariants on symplectic manifolds with S^1 -actions.

1. INTRODUCTION

This paper grew out of an attempt to understand when a circle action on a symplectic manifold (M, ω) gives rise to an **essential** (i.e. noncontractible) loop in the symplectomorphism group $\text{Symp}(M, \omega)$. Since nonHamiltonian loops have nonzero image under the Flux homomorphism

$$\text{Flux} : \pi_1(\text{Symp}(M, \omega)) \longrightarrow H^1(M, \mathbb{R}),$$

it suffices to restrict attention to circle subgroups of the Hamiltonian group $\text{Ham} := \text{Ham}(M, \omega)$. These actions are generated by functions $K : M \longrightarrow \mathbb{R}$, the **moment map**. We shall always assume that K is normalized, i.e. that $\int_M K \omega^n = 0$, and shall denote the corresponding circle action by Λ_K .

The main tool that we shall use is the Seidel representation of $\pi_1(\text{Ham})$ in the group of automorphisms of the quantum cohomology ring $QH^*(M)$ of the manifold (M, ω) . This is very difficult to calculate in general; some of the reasons for this are explained in the examples in §5.2. However, we make some progress in the case

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that the fixed point set M^{S^1} has a **semifree** component F , i.e. a component F so that the action is semifree on some neighborhood of F . (Recall that a circle action is semifree if the stabilizer of every point is trivial or the whole circle.) If the action is semifree on the whole manifold, we calculate in Theorem 1.14 the leading order term of the Seidel automorphism on quantum homology. This provides the technical basis for Gonzalez’s proof [6] that, if in addition all the fixed points are isolated, then the manifold (M, ω) has the same quantum cohomology as a product of 2-spheres.

Another interesting special case is when the manifold is toric and the maximal fixed component of Λ_K corresponds to one of the facets of the moment polytope. In this case, our results throw light on the multiplicative relations in the small quantum cohomology ring of M for arbitrary toric manifolds, though we can calculate them only in the Fano case: see §5.1.

We shall first state our results on the Hamiltonian group and then discuss properties of the Seidel representation.

1.1. Results on the Hamiltonian group. We get most information when one of the components on which K is a maximum or minimum is semifree.

Theorem 1.1. *Suppose that the Hamiltonian circle Λ_K has a semifree maximal or minimal fixed point component. Then Λ_K is essential in $\text{Ham}(M, \omega)$.*

This generalizes the result of McDuff–Slimowitz [17] stating that Λ_K is essential if the action is semifree. It follows from Theorem 1.9 (i) below, which calculates the leading order term of the Seidel element in the case when the maximal fixed point component is semifree. Because we are dealing with the maximal component, the proof is elementary; and the result itself is well known to experts, though as far as we know it is not formally stated in the literature. In contrast our later results are new. Moreover their proofs are considerably harder. Because most S^1 -manifolds do not admit invariant, ω -compatible and semipositive almost complex structures J (i.e. they are not symplectically NEF), it is important to work with general symplectic manifolds and hence with the virtual moduli cycle. In §4.2 we prove new results about localization for Gromov–Witten invariants, that are familiar in the algebraic context but not in the symplectic case.

The proof of Theorem 1.1 extends to cases where the isotropy of an extremal fixed component F is small compared to the “size” of F . We can also deal with nonextremal fixed components provided that the points above F have sufficiently small isotropy.

To be precise, we need a few definitions. We say that a subset $N \subset M$ has **at most k -fold isotropy** if the stabilizer of every point in N has at most k components. For each fixed point x we denote by $m(x)$ the sum of the weights at x . Finally, given a fixed component F , after choosing an ω -compatible S^1 -invariant almost complex structure J , decompose the negative normal bundle of F as a sum of complex vector bundles $E_1 \oplus \cdots \oplus E_\ell$ with weights $-k_1, \dots, -k_\ell$, where $k_i \geq 1$. The associated **obstruction bundle** is the bundle

$$(1) \quad \mathcal{E} := (E_1 \otimes \mathbb{C}^{k_1-1}) \oplus \cdots \oplus (E_\ell \otimes \mathbb{C}^{k_\ell-1}).$$

Note that summands E_i with $k_i = 1$ do not appear in \mathcal{E} since $E_i \otimes \mathbb{C}^{k_i-1} = \{0\}$ in this case. We say that F is **homologically visible** if the positive weights along F are all $+1$, and the associated obstruction bundle has nonzero Euler class. We

denote by $e(\mathcal{E}) \in H_*(F)$ the Poincaré dual of this Euler class. In particular, if all the negative weights are -1 , $e(\mathcal{E}) = [F] \in H_{\dim F}(F)$.

Theorem 1.2. *Consider a Hamiltonian circle action Λ_K on a compact symplectic manifold (M, ω) with moment map K . Assume that Λ_K is inessential in $\text{Ham}(M, \omega)$. Let F be a homologically visible fixed component.*

- (i) *F cannot be the maximal fixed component.*
- (ii) *More generally, if every point in the superlevel set $\{K(x) > K(F)\}$ has at most twofold isotropy, then $K(F) = m(F) = 0$ and F is semifree.*

Even if the conditions above are not satisfied, the existence of a semifree fixed point component still gives some information, though this is hard to interpret unless one has some global information about the action. The reason is that, although semifree fixed components F with nonzero $K(F)$ or $m(F)$ always make nontrivial contributions to the Seidel element, these could in general be cancelled by something else. One special case is when the circle contracts in a compact Lie subgroup G of the diffeomorphism group. In this case, we can apply Theorem 1.3 in McDuff–Tolman [18] which states that any semifree fixed component F has a **reversor**, that is there is $g \in G$ which fixes F but reverses Λ in the sense that $g^{-1}\Lambda g = \Lambda^{-1}$. In the Hamiltonian case, a symplectomorphism g is a reversor if and only if $K \circ g = -K$. The following result is an immediate consequence.

Proposition 1.3. *Consider a Hamiltonian circle action Λ_K on a compact connected symplectic manifold (M, ω) with moment map K . Let F be a semifree component of the fixed point set. Let $G \subset \text{Ham}(M, \omega)$ be a compact Lie group which contains Λ_K . If Λ_K is inessential in G , then there is an element $g \in G$ such that $g(F) = F$ and $K \circ g = -K$. In particular, $K(F) = m(F) = 0$.*

Another special case is when the underlying manifold is toric and the circle Λ_K is a subgroup of the n -torus T that acts on M (where $\dim M = 2n$). We denote by $\Phi : M \rightarrow \mathfrak{t}^*$ the mean normalized moment map; the moment image is the polytope $\Delta = \Phi(M) \subset \mathfrak{t}^*$. Given a face $f \subset \Delta$ of dimension k , the preimage $\Phi^{-1}(f)$ represents a homology class on M of degree $2k$. If Λ is any circle in T , the moment map K for Λ is the composite of Φ with the associated projection.

Proposition 1.4. *Fix a symplectic toric manifold (M, ω, Φ) with moment image Δ . Consider a circle subgroup $\Lambda_K \subset T$ which is inessential in $\text{Ham}(M, \omega)$. Let F be a semifree fixed point component for this circle action. Then $K(F) = m(F) = 0$. Moreover, let f^+ and f^- be the largest faces of Δ whose minimum and maximum, respectively, are $\Phi(F)$. Then, $[\Phi^{-1}(f^+)] = [\Phi^{-1}(f^-)] \in H_*(M)$.*

This result does go further than Proposition 1.3 since there are toric manifolds M that admit circles Λ_K which are inessential in $\text{Ham}(M, \omega)$ but are essential in the natural compact Lie groups that act on M : see McDuff–Tolman [19].

Another tractable situation is when the action Λ_K has at most twofold isotropy. In this case, even if there is no semifree fixed point component, the homology of the manifold M has the symmetry properties that one would expect if the action had a reversor ϕ . (Again, this echoes a result in McDuff–Tolman [18] which states that if a Hamiltonian circle action is inessential in a compact Lie subgroup G of $\text{Ham}(M, \omega)$, then it can be reversed.) Here is a precise statement.

Proposition 1.5. *Consider a Hamiltonian circle action Λ_K on a compact symplectic manifold (M, ω) with moment map K . Assume that Λ_K is inessential in $\text{Ham}(M, \omega)$ and that Λ_K has at most twofold isotropy.*

- (i) *For all $\mu \in \mathbb{R}$, a homology class of M can be represented in the sublevel set $\{K(x) < \mu\}$ if and only if it can be represented in the superlevel set $\{K(x) > -\mu\}$.*
- (ii) *For any connected component $N \subset M^{\mathbb{Z}/(2)}$, any integers j and n , and any $\mu \in \mathbb{R}$,*

$$\bigoplus_{\substack{F \subset N \\ K(F)=\mu, m(F)=n}} H_{j-\alpha_F}(F) \cong \bigoplus_{\substack{F' \subset N, \\ K(F')=-\mu, m(F')=-n}} H_{j-\beta_{F'}}(F'),$$

where the sums are over fixed components, α_F is the Morse index of F with respect to K , and $\beta_{F'}$ is the Morse index of F' with respect to $-K$.

Let F_{\max} and F_{\min} denote the maximal and minimal components of K , and define $K_{\max} = K(F_{\max})$, $K_{\min} = K(F_{\min})$, $m_{\max} = m(F_{\max})$, and $m_{\min} = m(F_{\min})$. The proposition above immediately implies that M is symmetrical with respect to K , in the sense that $K_{\max} = -K_{\min}$, $m_{\max} = -m_{\min}$, and $H_i(F_{\max}) = H_i(F_{\min})$ for all i . Similarly, every component N of $M^{\mathbb{Z}/(2)}$ is symmetrical with respect to K in the same sense.

More generally one can show that the moment image is not too skew whenever the isotropy weights are not too large. Given two fixed point components A and B , let $q = q(A, B)$ denote the largest integer so that A and B lie in the same component of $M^{\mathbb{Z}/(q)}$.

Proposition 1.6. *Consider a Hamiltonian circle action Λ_K on a compact symplectic manifold (M, ω) with moment map K . Assume that Λ_K is inessential in $\text{Ham}(M, \omega)$. Then there exists some sequence of fixed components $F_{\max} = F_0, F_1, \dots, F_j = F_{\min}$ with $K(F_i) \neq K(F_{i-1})$ for all i so that*

$$K_{\max} \geq \sum_{i=1}^j \frac{|K(F_{i-1}) - K(F_i)|}{q(F_{i-1}, F_i)}.$$

Moreover, we can choose the sequence so either the inequality above is strict, or

$$m_{\max} = \sum_{i=1}^j \frac{(m(F_{i-1}) - m(F_i))}{q(F_{i-1}, F_i)} \cdot \frac{K(F_{i-1}) - K(F_i)}{|K(F_{i-1}) - K(F_i)|}.$$

Note that if M has at most k -fold isotropy, then for any sequence of fixed point components $F_{\max} = F_0, F_1, \dots, F_j = F_{\min}$ with $K(F_{i-1}) \neq K(F_i)$ for all i , $\sum \frac{|K(F_{i-1}) - K(F_i)|}{q(F_{i-1}, F_i)} \geq \frac{K_{\max} - K_{\min}}{k}$. Moreover, the inequality is strict unless F_{\max} and F_{\min} (and in fact all the F_i) lie in the same component of $M^{\mathbb{Z}/(k)}$, and $K(F_{i-1}) > K(F_i)$ for all i . In this case, $\sum \frac{(m(F_{i-1}) - m(F_i))}{q(F_{i-1}, F_i)} \cdot \frac{K(F_{i-1}) - K(F_i)}{|K(F_{i-1}) - K(F_i)|} = \frac{m_{\max} - m_{\min}}{k}$. Therefore, the proposition has the following corollary.

Corollary 1.7. *Suppose in the situation of Proposition 1.6 that M has at most k -fold isotropy. Then, after possibly reversing the circle action,*

$$K_{\max} \leq |K_{\min}| \leq (k-1)K_{\max},$$

and the second inequality is strict unless $m_{\min} = (k-1)|m_{\max}|$, and F_{\max} and F_{\min} lie in the same component of $M^{\mathbb{Z}/(k)}$.

Observe finally that the inequality in Proposition 1.6 is sharp: just consider the action on \mathbb{CP}^k given by $[z_0 : z_1 : z_2 : \dots : z_k] \mapsto [\lambda^k z_0 : \lambda z_1 : z_2 : \dots : z_k]$.

Similar but more complicated statements can be made about the other fixed point components: see Prop 3.11.

1.2. Results on the Seidel representation. We now state our main results on the Seidel representation and use them to deduce Theorems 1.1 and 1.2, and Propositions 1.4 and 1.5.

The Seidel representation is a quantum version of a classical homomorphism defined by Weinstein [25] using the action functional. Let $H_*^S(M) := H_*^S(M; \mathbb{Z})$ denote the spherical homology of M . Let $I_\omega, I_c : H_2^S(M) \rightarrow \mathbb{R}$ denote the homomorphisms induced by evaluating the classes $[\omega]$ and $c_1 = c_1(TM) \in H^2(M, \mathbb{Z})$. Weinstein's homomorphism $\mathcal{A}_\omega : \pi_1(\text{Ham}(M)) \rightarrow \mathbb{R}/(\text{im } I_\omega)$ takes the circle Λ_K to the value $K(x)$ of the generating moment map at any critical point.¹ As we show in §2.3 this extends to take the weights into account.

Lemma 1.8. *Let (M, ω) be a compact symplectic manifold. There is a homomorphism*

$$\mathcal{A}_{\omega, c} : \pi_1(\text{Ham}(M, \omega)) \rightarrow \mathbb{R} \oplus \mathbb{Z}/\text{im}(I_\omega \oplus I_c)$$

whose value at a Hamiltonian circle action Λ_K is $[K(x), -m(x)]$, where x is any critical point of K .

The Seidel representation

$$\mathcal{S} : \pi_1(\text{Ham}(M, \omega)) \rightarrow \text{QH}_{\text{ev}}(M; \Lambda)^\times$$

is a lift of $\mathcal{A}_{\omega, c}$ to the group of even units $\text{QH}_{\text{ev}}(M; \Lambda)^\times$ of the quantum homology ring $\text{QH}_*(M) = \text{QH}_*(M; \Lambda) := H_*(M) \otimes \Lambda$ of M : see [23, 9, 13]. Here, following [15], we use coefficients $\Lambda := \Lambda^{\text{univ}}[q, q^{-1}]$ where q is a variable of degree 2 and Λ^{univ} is a generalized Laurent series ring in a variable t of degree 0:

$$\Lambda^{\text{univ}} := \left\{ \sum_{\kappa \in \mathbb{R}} r_\kappa t^\kappa \mid r_\kappa \in \mathbb{Q}, \#\{\kappa > c \mid r_\kappa \neq 0\} < \infty, \forall c \in \mathbb{R} \right\}.$$

We shall order the elements $\sum_{d, \kappa} a_{d, \kappa} \otimes q^d t^\kappa$ in $\text{QH}_*(M; \Lambda)$ by the valuation²

$$v\left(\sum_{d, \kappa} a_{d, \kappa} \otimes q^d t^\kappa\right) = \max\{\kappa \mid \exists d : a_{d, \kappa} \neq 0\}.$$

For more details see §2.3.

The image $\mathcal{S}(\Lambda) \in \text{QH}_{\text{ev}}(M; \Lambda)^\times$ of the Hamiltonian loop Λ is called the **Seidel element** of Λ . It has degree $\dim M$ and gives rise to a degree preserving automorphism of $\text{QH}_*(M)$ by quantum multiplication:

$$\mathcal{S}(\Lambda)(a) := \mathcal{S}(\Lambda) * a.$$

Thus $\mathcal{S}(\Lambda) = \mathcal{S}(\Lambda)(\mathbb{1})$ where $\mathbb{1}$ denotes the unit $[M]$ in $\text{QH}_*(M)$. Our results are based on a partial calculation of $\mathcal{S}(\Lambda_K)$.

¹This homomorphism itself contains quite a bit of information: see for example McDuff–Tolman [19].

²One must treat this ordering with some care. Although $v(a * b) \leq v(a) + v(b)$ for all $a, b \in \text{QH}_*(M)$ with equality only if the usual intersection product of the highest order terms is nonzero, in the case when this intersection product is zero the term of highest order in $a * b$ may not be equal to the product of the highest order terms in a and b .

If a circle action Λ is inessential, then $\mathcal{S}(\Lambda) = \mathbb{1}$, so $\mathcal{S}(\Lambda)(a) = a$ for all homology classes a . Therefore, if we write

$$\mathcal{S}(\Lambda)(a) = \sum_{d,\kappa} c_{d,\kappa} \otimes q^d t^\kappa,$$

there are potentially two techniques to show that Λ is essential. First, we can show that $c_{d,\kappa} \neq 0$ for some pair $(d, \kappa) \neq (0, 0)$; this is the approach we take for homologically visible fixed points. Second, we can find a homology class b so that $a \cdot b \neq 0$ but that $c_{0,0} \cdot b = 0$; this is the approach that we take in all other cases.

The following theorem is proved in Section 3.2. It is a considerable generalization of Proposition 7.11 in Seidel [23], which applies only in the case of monotone (M, ω) and does not discuss the structure of the higher order terms. By analogy with the algebraic case, we say that the almost complex manifold (M, J) is Fano (resp. NEF) if there are no J -holomorphic spheres in classes B with $c_1(B) \leq 0$ (resp. $c_1(B) < 0$).

Theorem 1.9. *Consider a Hamiltonian circle action Λ_K on a compact symplectic manifold (M, ω) with normalized moment map K . Assume that the maximal fixed point component F_{\max} is semifree. Then:*

(i)

$$\mathcal{S}(\Lambda_K) = [F_{\max}] \otimes q^{-m_{\max}} t^{K_{\max}} + \sum_{B \in H_2^S(M): \omega(B) > 0} a_B \otimes q^{-m_{\max} - c_1(B)} t^{K_{\max} - \omega(B)},$$

where a_B is the contribution from the section class $\sigma_{\max} + B$. Moreover, if $a_B \neq 0$ then $\deg(a_B) = \dim F_{\max} + 2c_1(B)$.

(ii) If (M, J) is Fano (resp. NEF) for some S^1 -invariant ω -compatible almost complex structure J then $a_B = 0$ unless $c_1(B) > 0$ (resp. $c_1(B) \geq 0$).

(iii) Assume that (M, J) is NEF for some S^1 -invariant ω -compatible almost complex structure J . If $2c_1(B') \geq \text{codim } F_{\max}$ for every J -holomorphic sphere B' which intersects F_{\max} then all the lower order terms vanish. If the latter hypothesis holds except for spheres which lie in F_{\max} itself, then $a_B = 0$ unless $2c_1(B) < \text{codim } F_{\max}$ and B lies in the image of the spherical homology $H_2^S(F_{\max})$.

The last sentence in part (i) of Theorem 1.9 expresses the fact that $\mathcal{S}(\Lambda_K)$ preserves degree. If $a_B \neq 0$ then

$$\deg(a_B \otimes q^{-m_{\max} - c_1(B)} t^{K_{\max} - \omega(B)}) = \deg a_B - 2m(F_{\max}) - 2c_1(B) = \dim M.$$

Moreover, $\dim F_{\max} = \dim M + 2m(F_{\max})$ because F_{\max} is semifree. Therefore

$$\deg a_B = \dim F_{\max} + 2c_1(B).$$

This theorem gives the most information when $\text{codim } F_{\max} = 2$, for example in the case of a circle action on a toric variety that fixes one facet. If, in addition (M, ω, J) is Fano for some S^1 -invariant ω -compatible J , then, by part (ii) all the lower order terms vanish. That is, $\mathcal{S}(\Lambda_K) = [F_{\max}] \otimes q^{-m_{\max}} t^{K_{\max}}$. In §3.2 we shall give a more precise description of the lower order terms in $\mathcal{S}(\Lambda)$. These remarks have consequences for the structure of the quantum cohomology of toric manifolds that are explained in §5.1.

Example 1.10. Consider the rotation of S^2 generated by the height function K and let $A = [S^2]$. Then $\mathcal{S}(\Lambda_K) = [pt] \otimes q t^{\omega(A)/2}$.

Proof of Theorem 1.1. If Λ_K is inessential in $\text{Ham}(M, \omega)$, then $\mathcal{S}(\Lambda_K) = \mathbb{1}$. Hence Theorem 1.1 follows immediately from the first claim of Theorem 1.9. \square

Proof of Proposition 1.4. Pick any $x \in F$. There is a neighborhood of x and an isomorphism of T with $(S^1)^n$ so that the action of T is equivariantly symplectomorphic to the standard action of $(S^1)^n$ on \mathbb{C}^n . In these coordinates, the action of Λ_K on \mathbb{C}^n is given by $\lambda z = (\lambda^{m_1} z_1, \dots, \lambda^{m_n} z_n)$, where m_1, \dots, m_n are the weights at x .

For $1 \leq i \leq n$, let D_i be the facet of Δ which corresponds to $z_i = 0$. Let $\eta_i \in \ell$ denote the outward primitive normal vector to D_i , where $\ell \subset \mathfrak{t}$ is the integral lattice. Note that $K_i := \langle \eta_i, \Phi(\cdot) \rangle$ is the moment map for a circle action Λ_i , and that $\Phi^{-1}(D_i)$ is a semifree maximum for this action. By Theorem 1.9, $\mathcal{S}(\Lambda_i) = y_i \otimes q t^{\eta_i(D_i)}$, where $y_i = [\Phi^{-1}(D_i)] +$ lower order terms. Since Λ_K is inessential, $\mathcal{S}(\Lambda_K) = \mathbb{1}$. On the other hand, by looking at the action near the fixed point x one sees that $\Lambda_K = \prod \Lambda_i^{-m_i}$. Therefore

$$\prod [y_i]^{-m_i} \otimes q^{m_i} t^{-m_i \eta_i(D_i)} = \mathbb{1},$$

where the product is taken in $\text{QH}_*(M; \Lambda)$.

Let $m_i = 1$ for $1 \leq i \leq r$, $m_i = -1$ for $r < i \leq r + s$, and $m_i = 0, i > r + s$. Then

$$y_1 * \dots * y_r \otimes q^{-r} t^{\eta_1(D_1) + \dots + \eta_r(D_r)} = y_{r+1} * \dots * y_{r+s} \otimes q^{-s} t^{\eta_{r+1}(D_{r+1}) + \dots + \eta_{r+s}(D_{r+s})}$$

In particular, the highest order terms must agree. Since $D_1 \cap \dots \cap D_r = f^+$ and $D_{1+r} \cap \dots \cap D_{r+s} = f^-$, we have $[\Phi^{-1}(D_1)] \cap \dots \cap [\Phi^{-1}(D_r)] = [\Phi^{-1}(f^+)]$ and $[\Phi^{-1}(D_{1+r})] \cap \dots \cap [\Phi^{-1}(D_{r+s})] = [\Phi^{-1}(f^-)]$. Since these intersections are nontrivial, the highest order terms in the quantum product of the corresponding y_i are given by these intersections. The result follows. \square

Remark 1.11. More generally, let F be a fixed component of any inessential Hamiltonian loop $\Lambda_K \subset T$. Let (m_1, \dots, m_n) be the weights at $x \in F$ with corresponding facets D_i . Define homology classes in M by $X^+ = \cap_{m_i > 0} [\Phi^{-1}(D_i)]^{m_i}$ and $X^- = \cap_{m_i < 0} [\Phi^{-1}(D_i)]^{-m_i}$. (Here, we are taking the ordinary cap product in homology.) If both X^+ and X^- are nonzero, then, by an argument similar to the one above, $K(F) = m(F) = 0$ and $X^+ = X^-$.

1.2.1. Semifree actions and canonical bases for homology. For a general action our methods do not give any information about $\mathcal{S}(\Lambda_K)(a) := \mathcal{S}(\Lambda_K) * a$ for $a \neq \mathbb{1}$. However, when the action is semifree, it is possible to describe the top order term in $\mathcal{S}(\Lambda_K)(a)$ for any $a \in H_*(M)$. This formula is best written in terms of some canonical bases $\{c_i^-\}$ and $\{c_i^+\}$ for $H_*(M)$.

Before explaining this, we introduce more notation. Given $\mu \in \mathbb{R}$, define

$$\begin{aligned} M_\mu &:= K^{-1}([\mu, \infty)), & M_{>\mu} &:= K^{-1}(\mu, \infty)), \\ M^\mu &:= K^{-1}((-\infty, \mu]), & M^{<\mu} &:= K^{-1}((-\infty, \mu)). \end{aligned}$$

The inclusions $M_\mu \rightarrow M$ and $M^\mu \rightarrow M$ induce maps $H_*(M_\mu) \rightarrow H_*(M)$ and $H_*(M^\mu) \rightarrow H_*(M)$ in rational homology. We call the images of these maps $F_\mu H_*(M)$ and $F^\mu H_*(M)$, respectively.

We now give a brief review of equivariant cohomology: Let S^1 act on a space N . The equivariant cohomology $H_{S^1}^*(N)$ of N is defined to be the cohomology of the

total space

$$N_{S^1} := S^\infty \times_{S^1} N$$

of the universal N -bundle over the classifying space $BS^1 = \mathbb{CP}^\infty$. Thus $H_{S^1}^*(N)$ is a module over $H_{S^1}^*(pt) \cong H^*(\mathbb{CP}^\infty)$, which is a polynomial ring with one generator u of degree 2. Moreover, there is a natural map from $H_{S^1}^*(N)$ to $H^*(N)$, given by restricting to any fiber.

If S^1 acts trivially on F there is a natural identification $H_{S^1}^*(F) = H_{S^1}^*(pt) \otimes H^*(F)$. Given $\tilde{Y} \in H_{S^1}^*(F)$, we say that the **degree** of \tilde{Y} in $H_{S^1}^*$ is j if j is the smallest integer such that

$$\tilde{Y} \in \bigoplus_{i=0}^j H_{S^1}^i(pt) \otimes H^*(F).$$

We now explain a procedure for producing a natural set of generators for the equivariant cohomology of M given a set of generators for the cohomology of each fixed component.

Lemma 1.12. *Let S^1 act on a compact symplectic manifold (M, ω) with moment map K . Let $F \subset M$ be any fixed component of index α ; let $e_F^- \in H_{S^1}^\alpha(F)$ be the equivariant Euler class of the negative normal bundle to F . Given any cohomology class $Y \in H^i(F)$, there exists a unique cohomology class $\tilde{Y}^+ \in H_{S^1}^{i+\alpha}(M)$ so that*

- (a) *The restriction of \tilde{Y}^+ to $M^{<K(F)}$ vanishes,*
- (b) *$\tilde{Y}^+|_F = Y \cup e_F^-$, and*
- (c) *the degree of $\tilde{Y}^+|_{F'}$ in $H_{S^1}^*(pt)$ is less than the index $\alpha_{F'}$ of F' for all fixed components $F' \neq F$.*

Moreover, these classes generate $H_{S^1}^*(M)$ as a $H_{S^1}^*(pt)$ module.

We can use this lemma, which we prove in section 4.1.1, to create a set of generators for the homology of M . Let F be a fixed component, and let α_F and β_F denote the index of F with respect to K and $-K$, respectively. Given a homology class $c \in H_i(F)$, we define the **upwards extension** $c^+ \in H_{i+\beta_F}(M)$ as follows:

- Let $Y \in H^{\dim F - i}(F)$ be the Poincaré dual to c .
- Let $\tilde{Y}^+ \in H_{S^1}^{\dim F + \alpha_F - i}(M)$ be the unique equivariant cohomology class which satisfies the conditions of Lemma 1.12.
- Let $Y^+ \in H^{\dim F + \alpha_F - i}(M)$ be the restriction of \tilde{Y}^+ to ordinary cohomology.
- Let $c^+ \in H_{i+\dim M - \dim F - \alpha_F}(M) = H_{i+\beta_F}(M)$ be the Poincaré dual to Y^+ .

Note that, by construction, c^+ lies in $F_{K(F)}H_*(M)$. The **downwards extension** $c^- \in H_{i+\alpha_F}(M)$, which lies in $F^{K(F)}H_*(M)$ is defined analogously; simply replace K by $-K$.

Since the classes \tilde{Y}^+ generate $H_{S^1}^*(M)$ as a $H_{S^1}^*(pt)$ module and the restriction $H_{S^1}^*(M) \rightarrow H^*(M)$ is surjective, the classes Y^+ generate $H^*(M)$ as a (rational) vector space. Hence, the classes c^+ (or, alternatively, the classes c^-) generate $H_*(M)$ as a vector space.

When the action is semifree, the classes c^+ and c^- have a nice geometric description. Assume that c can be represented by an i -dimensional submanifold $C \subset F$. By Lemma 4.6, if g_J is the metric associated to a generic S^1 -invariant ω -compatible

almost complex structure J and we choose C generically, the stable manifold $W^s(C)$ is an $(i + \beta_F)$ -dimensional pseudocycle. (See section 4.1.2.) Hence, it represents a homology class $[W^s(C)] \in H_{i+\beta_F}(M)$. By Proposition 4.8, $[W^s(C)] = c^+$. Similarly, c^- is represented by the unstable manifold $[W^u(C)]$.

Remark 1.13. We may define an automorphism $\mathcal{D}_K : H_*(M) \longrightarrow H_*(M)$ by

$$\mathcal{D}_K(c^-) = c^+, \quad c \in H_*(M^{S^1}).$$

For example, if $c = [F_{\max}] \in H_*(F_{\max})$ is the maximal fixed point set of K , then $c^- = \mathbb{1}$ while $c^+ = [F_{\max}]$. Therefore

$$\mathcal{D}_K(\mathbb{1}) = [F_{\max}].$$

If K is Morse then \mathcal{D}_K can be interpreted as a form of duality. If $\{c_i\}$ is given by the set of critical points of K , then the bases $\{c_i^-\}$ and $\{c_i^+\} = \{\mathcal{D}_K(c_i^-)\}$ are dual with respect to the intersection pairing. Although it is tempting to think that \mathcal{D}_K is an involution, in fact the correct relation is $\mathcal{D}_{-K} \circ \mathcal{D}_K = \mathbb{1}$. \square

The following theorem is proved in section 3.3.

Theorem 1.14. *Let S^1 act semifreely on a compact symplectic manifold (M, ω) . Let F be a component of the fixed point set, and choose a homology class $c \in H_*(F)$. Then*

$$(2) \quad \mathcal{S}(\Lambda_K)(c^-) = c^+ \otimes q^{-m(F)} t^{K(F)} + \sum_{B \in H_2^S(M) : \omega(B) > 0} a_B \otimes q^{-m(F) - c_1(B)} t^{K(F) - \omega(B)}.$$

Moreover if $a_B \neq 0$ then $\deg a_B = \deg c^+ + 2c_1(B)$.

Since every element $a \in H_*(M)$ can be written as a linear combination of such c^- , this theorem gives the leading order term of $\mathcal{S}(\Lambda_K)(a)$ for every $a \in H_*(M)$.

The last claim of the theorem follows from the fact that \mathcal{S} preserves degree. This implies that if $a_B \neq 0$, then

$$\deg(a_B) - 2m(F) - 2c_1(B) = \deg(c^-).$$

Since the action is semifree, $m(F)$ is the number of positive weights minus the number of negative weights. But the degree of c^+ is the degree of c plus twice the number of positive weights, and the degree of c^- is the degree of c plus twice the number of negative weights. Hence $\deg(a_B) = \deg(c^+) + 2c_1(B)$.

Example 1.15. Think of \mathbb{CP}^2 as the manifold obtained from the closed unit ball in \mathbb{C}^2 by identifying its boundary to a complex line via the Hopf map, and consider the action

$$(z_1, z_2) \mapsto (\lambda^{-1} z_1, \lambda^{-1} z_2).$$

Then $K(z_1, z_2) = \pi(c - |z_1|^2 - |z_2|^2)$ where $c = 2/3$, $F_{\max} = \{pt\}$ and all critical points are semifree. Since $c_1(L) = 3$ where $L = [\mathbb{CP}^1]$, there can be no lower order terms in the formula for $\mathcal{S}(\Lambda_K)(a)$ since the dimensional condition can never be satisfied. Hence, since $\omega(L) := \pi$, we find that $\mathcal{S}(\Lambda_K)$ acts by:

$$\mathbb{1} \mapsto [pt] \otimes q^2 t^{2\pi/3}, \quad L \mapsto \mathbb{1} \otimes q^{-1} t^{-\pi/3}, \quad [pt] \mapsto L \otimes q^{-1} t^{-\pi/3},$$

which is consistent with the formula $\mathcal{S}(\Lambda_K)(a) = \mathcal{S}(\Lambda_K) * a$. The above results also agree with the formulas found in [16] §4 for rotations of the one point blow up of

\mathbb{CP}^2 : see Example 5.6. In this example we shall also see that although Theorem 1.9 implies that there are no lower order terms in the Seidel element $\mathcal{S}(\Lambda)$ itself if (M, ω) is Fano and F_{\max} has codimension 2, there may be lower order terms in $\mathcal{S}(\Lambda)(a)$ for such actions.

1.2.2. Actions with at most twofold isotropy. We can also obtain some information about $\mathcal{S}(\Lambda_K)(a)$, though considerably less than before, when Λ_K acts with at most twofold isotropy. Throughout the following discussion we denote by \cdot_Y the intersection pairing $H_k(Y) \times H_{m-k}(Y) \rightarrow \mathbb{Q}$ on the homology of an oriented m -dimensional manifold Y . For convenience we set $a \cdot_Y b = 0$ whenever the dimensional condition $\deg(a) + \deg(b) = m$ is not satisfied.

The following theorem is proved in section 3.4.

Theorem 1.16. *Consider a Hamiltonian circle action Λ_K on a compact symplectic manifold (M, ω) with at most twofold isotropy. Let F and F' be components of the fixed point set. Choose homology classes $c \in H_*(F)$ and $c' \in H_*(F')$, and write*

$$\mathcal{S}(\Lambda_K)(c^-) = \sum_{d, \kappa} c_{d, \kappa} \otimes q^d t^\kappa.$$

If $K(F') \leq -K(F)$, then $c_{0,0} \cdot_M (c')^- = 0$ unless $K(F) = -K(F')$, $m(F) = -m(F')$, and F and F' lie in the same component of $M^{\mathbb{Z}/(2)}$.

Proof of Proposition 1.5. Let F and F' be components of the fixed point set so that $K(F') \leq -K(F)$. Choose homology classes $c \in H_*(F)$ and $c' \in H_*(F')$. Since Λ_K is inessential, $\mathcal{S}(\Lambda_K)(c^-) = c^- \otimes \mathbb{1}$. Hence, by Theorem 1.16 $c^- \cdot_M (c')^- = 0$ unless $K(F') = -K(F)$, $m(F') = -m(F)$, and F and F' lie in the same component of $M^{\mathbb{Z}/(2)}$.

For any $\mu \in \mathbb{R}$, $F^\mu H_*(M)$ is generated by elements c^- , where $c \in H_*(F)$ and F is a fixed component with $K(F) \leq \mu$. The paragraph above implies that every such c^- lies in $F_{-\mu} H_*(M)$. Hence $F^\mu H_*(M) \subseteq F_{-\mu} H_*(M)$. Similarly, applying the theorem to the moment map $-K$, $F^\mu H_*(M) \supseteq F_{-\mu} H_*(M)$. Hence $F^\mu H_*(M) = F_{-\mu} H_*(M)$. This proves the first claim.

Since both K and $-K$ are perfect Morse functions, both $H_*(M_\mu) \rightarrow H_*(M)$ and $H_*(M^\mu) \rightarrow H_*(M)$ are injections. Hence, the arguments above imply that

$$H_j(M^\mu, M^{<\mu}) = H_j(M_{-\mu}, M_{>-\mu}).$$

By the Thom isomorphism theorem, this is equivalent to

$$\bigoplus_{K(F)=\mu} H_{j-\alpha_F}(F) = \bigoplus_{K(F')=-\mu} H_{j-\beta_{F'}}(F'),$$

where the sums are over fixed components, α_F is the index of F with respect to K , and $\beta_{F'}$ is the index of F' with respect to $-K$.

Now suppose that F and F' lie in different components of the isotropy submanifold $M^{\mathbb{Z}/(2)}$ and satisfy $K(F) = -K(F') = \mu$. Consider $c \in H_i(F)$ and $c' \in H_*(F')$. We saw above that $c^- \cdot_M (c')^- = 0$. Hence, the F' component of the image of c under the isomorphism above must be zero. Hence, the isomorphism above is still an isomorphism when restricted to any component of $M^{\mathbb{Z}/(2)}$. A similar argument applies if $m(F) \neq -m(F')$. \square

Finally, let us consider the contribution of a homologically visible component. The following theorem is also proved in §3.4.

Theorem 1.17. *Consider a Hamiltonian circle action Λ_K on a compact symplectic manifold. Let F be a fixed component. Assume that all the positive weights at F are $+1$. Suppose further that the superlevel set $\{K(x) > K(F)\}$ has at most twofold isotropy. Choose a homology class $c \in H_*(F)$, and write*

$$\mathcal{S}(\Lambda_K)(c^-) = \sum_{d, \kappa} c_{d, \kappa} \otimes q^{-m(F)+d} t^{K(F)+\kappa}.$$

Then $c_{0,0} \in F_{K(F)} H_*(M)$. Moreover, for any $c' \in H_*(F)$,

$$c_{0,0} \cdot_M (c')^- = (e(\mathcal{E}) \cap_F c) \cdot_F c'$$

where $e(\mathcal{E})$ denotes the Poincaré dual of the Euler class of the obstruction bundle $\mathcal{E} \rightarrow F$. (See equation (1).)

Proof of Theorem 1.2. Let F be a homologically visible fixed component and assume that every point in the superlevel set $\{K(x) > K(F)\}$ has at most twofold isotropy. Apply Theorem 1.17 with $c = \mathbb{1}_F \in H_*(F)$ and $c' \in H_*(F)$ chosen so that $e(\mathcal{E}) \cdot_F c' = k \neq 0$. Then $c_{0,0} \cdot_M (c')^- = e(\mathcal{E}) \cdot_F c' = k \neq 0$. Therefore the coefficient $c_{0,0}$ of $q^{-m(F)} t^{K(F)}$ in $\mathcal{S}(\Lambda_K)((\mathbb{1}_F)^-)$ is nonzero. Since Λ_K is inessential, $\mathcal{S}(\Lambda_K)((\mathbb{1}_F)^-) = (\mathbb{1}_F)^-$. Therefore $K(F) = m(F) = 0$ and $c_{0,0} = (\mathbb{1}_F)^-$, so $e(\mathcal{E}) = (\mathbb{1}_F)$. This proves (ii). Item (i) is simply a special case. \square

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2. QUANTUM HOMOLOGY AND THE SEIDEL REPRESENTATION

This section reviews the necessary background material. The main geometric idea behind our results, symplectic bundles over the two sphere, is explained in §2.1. We review (small) quantum homology in §2.2 to fix notational conventions, and then describe the Seidel representation in §2.3.

2.1. Symplectic bundles over the two sphere. Throughout we shall use the following notational/sign conventions. If $H_t, 0 \leq t \leq 1$, is a (time dependent) Hamiltonian then we define the corresponding vector field X_H by the identity

$$(3) \quad \omega(X_H, \cdot) = -dH_t.$$

Thus $X_H = J(\text{grad } H_t)$, where J is an ω -compatible almost complex structure and the gradient is taken with respect to the metric g_J given by $g_J(x, y) = \omega(x, Jy)$. As an example, consider the unit sphere S^2 in \mathbb{R}^3 , oriented via stereographic projection from the north pole.³ Then its area form is $dx_3 \wedge d\theta$ and the vector field X_K generated by the normalized height function $K = 2\pi x_3$ is $X_K = 2\pi \partial_\theta$. Thus the corresponding flow is the anticlockwise rotation of S^2 about the axis from the south to the north pole. Note that this flow is positive (i.e. anticlockwise) at the south pole s (the minimum of K) and negative at the north pole n (the maximum of K), which agrees with the usual conventions for defining the **moment map**.

Consider the locally trivial bundle $P_\Lambda \rightarrow S^2$ constructed by using $\Lambda = \{\phi_t\} \in \pi_1(\text{Ham}(M))$ as a clutching function:

$$(4) \quad P_\Lambda = (D_0 \times M) \cup (D_\infty \times M) / \sim, \quad \text{where } (e^{2\pi i t}, \phi_t(x))_0 \sim (e^{2\pi i t}, x)_\infty.$$

Here we are thinking of D_0 as the closed unit disc centered at 0 in the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ and of D_∞ as another copy of this disc, embedded in $S^2 = \mathbb{C} \cup \{\infty\}$ via the orientation reversing map $r e^{i\theta} \mapsto r^{-1} e^{i\theta}$. Correspondingly, we denote the fibers over $0, \infty$ by M_0, M_∞ . Note that our definition of P_Λ agrees with that in [16] but differs in orientation from the convention used in [9, 13].

The fact that Λ is Hamiltonian implies that there is a closed 2-form Ω on P_Λ extending the fiberwise symplectic forms: see [23] or [14] Chapter 6 for example. Conversely, every pair consisting of a smooth bundle $\pi : P \rightarrow S^2$ with fiber M together with a closed 2-form Ω on P that is nondegenerate on each fiber arises in this way from a loop in $\text{Ham}(M, \omega)$. By adding to Ω the pullback of a suitable area form on the base, we may assume that Ω is nondegenerate. Any such symplectic extension of the fiberwise forms will be called **ω -compatible**. The set of these forms is contractible. Note that each such Ω gives rise to a connection on P with Hamiltonian holonomy, whose horizontal distribution consists of the Ω -orthogonals to the fibers.

Each such triple (P, π, Ω) admits a contractible family $\mathcal{J}(P, \pi, \Omega)$ of Ω -compatible almost complex structures \tilde{J} such that $\pi : (P, \tilde{J}) \rightarrow (S^2, j_0)$ is holomorphic. Each $\tilde{J} \in \mathcal{J}(P, \pi, \Omega)$ preserves the tangent bundle to the fibers and hence also the horizontal distribution.

³This means that the vertical projection from the tangent space at the south pole to the (x_1, x_2) -plane preserves orientation. Hence this orientation is the opposite of its orientation as the boundary of the unit ball.

Now observe that the bundle $(P_\Lambda, \Omega) \longrightarrow S^2$ supports two canonical cohomology classes. The first is the **first Chern class of the vertical tangent bundle**

$$c_{\text{vert}} = c_1(TP_\Lambda^{\text{vert}}) \in H^2(P_\Lambda, \mathbb{Z}).$$

The second is the **coupling class**, which is the unique class $u_\Lambda \in H^2(P_\Lambda, \mathbb{R})$ such that

$$i^*(u_\Lambda) = [\omega], \quad u_\Lambda^{n+1} = 0,$$

where $i : M \longrightarrow P_\Lambda$ is the inclusion of a fiber.

Another important geometric fact about Hamiltonian bundles over S^2 is that they always have sections. A direct geometric argument shows that this is equivalent to saying that the map

$$\pi_1(\text{Ham}(M, \omega)) \longrightarrow \pi_1(M, x) : \quad \{\phi_t\} \mapsto \{\phi_t(x)\}$$

given by evaluation at the base point x is trivial. The latter statement follows from the proof of the Arnol'd conjecture or by the very existence of the Seidel representation: see [9]. Therefore, in particular, there always exists a **section class**, that is, a class $\sigma \in H_2^S(P, \mathbb{R})$ that projects onto the positive generator of $H_2(S^2, \mathbb{Z})$. We shall denote by $H_2^{\text{sec}}(P; \mathbb{Z})$ the affine subspace of $H_2(P; \mathbb{Z})$ consisting of such section classes.

Circle actions. We now assume that Λ_K is a circle action with moment map K , and show how the ideas above simplify in this case. Let S^1 act on $S^3 \times M$ by the diagonal action

$$(z_1, z_2; x) \mapsto (e^{2\pi it} z_1, e^{2\pi it} z_2; \phi_t x).$$

We claim that P_Λ can be identified with the quotient $S^3 \times_{S^1} M$. To see this, write $[z_1, z_2; x]$ for the equivalence class containing the point $(z_1/r, z_2/r; x) \in S^3 \times M$, where $r^2 = |z_1|^2 + |z_2|^2$. In these coordinates,

$$D_0 \times M = \{[z, 1; x] : |z| \leq 1, x \in M\} \quad \text{and} \quad D_\infty \times M = \{[1, z; x] : |z| \leq 1, x \in M\}.$$

In particular, M_0 is the fiber at $[0 : 1] \in \mathbb{CP}^1 = S^3/S^1$, and M_∞ is the fiber at $[1 : 0]$. Both have natural identifications with M . Since the orientation on D_∞ was reversed, the gluing map is given by

$$[1, e^{-2\pi it}; x] \sim [e^{2\pi it}, 1; \phi_t x],$$

as required.

Let $\alpha \in \Omega^1(S^3)$ be the usual contact form on the unit sphere, normalized so that $d\alpha = \chi^*(\tau)$ where $\chi : S^3 \longrightarrow S^2$ is the Hopf map and τ is the standard area form on S^2 with total area 1. Given any $c \in \mathbb{R}$, the form $\omega + c d\alpha - d(K\alpha) \in \Omega^2(S^3 \times M)$ is closed and basic, and hence descends under the projection $\text{pr} : S^3 \times M \longrightarrow S^3 \times_{S^1} M$ to a closed two form Ω_c on P_Λ which extends the fiberwise symplectic form. Thus,

$$(5) \quad \Omega_c = \text{pr}_*(\omega + d((c - K)\alpha)) = \text{pr}_*(\omega + c d\alpha - d(K\alpha)).$$

If $c > \max K$, then Ω_c is symplectic. The coupling class is simply $[\Omega_0]$:

$$(6) \quad u_\Lambda = [\Omega_0] = [\text{pr}_*(\omega - d(K\alpha))].$$

Remark 2.1. Note that, if c_1 and $[\omega]$ are linearly dependent, then, since the S^1 -orbit of an arc going from the minimum to the maximum of K is a sphere on which both ω and c_1 are positive, (M, ω) must be monotone, that is, $I_c = \mu I_\omega$ for some $\mu > 0$.

Each fixed point x of the S^1 -action gives rise to a section of P

$$\sigma_x := S^3 \times_{S^1} \{x\} = D_0 \times \{x\} \cup D_\infty \times \{x\}.$$

We will sometimes write σ_F or σ_{\max} instead of σ_x , when $x \in F$ or $x \in F_{\max}$, respectively. Here are some useful facts about these sections.

Lemma 2.2. *If x is a fixed point of a Hamiltonian circle action Λ_K on a symplectic manifold (M, ω) , then*

$$c_{\text{vert}}(\sigma_x) = m(x) \quad \text{and} \quad u_\Lambda(\sigma_x) = -K(x).$$

Moreover, if B is the class of the sphere formed by the Λ -orbit of an arc from x to another fixed point y , then $B = \sigma_x - \sigma_y$.

Proof. The normal bundle of σ_x can be identified with a sum of holomorphic line bundles $L_i \rightarrow \mathbb{CP}^1$, one for each weight m_i at x . Moreover, $c_1(L_i) = m_i$. Thus $c_{\text{vert}}(\sigma_x) = m(x)$. Further, by Equation (6) $u_\Lambda(\sigma_x) = [\omega - d(K\alpha)](\sigma_x) = -K(x)$. This proves the first claim.

Using the sign conventions explained at the beginning of §2.1, one finds by an easy calculation that $\omega(B) = K(y) - K(x) = u_\Lambda(\sigma_x - \sigma_y) = \omega(\sigma_x - \sigma_y)$. This identity holds for all Λ -invariant symplectic forms ω' on M . But, after averaging, any closed 2-form sufficiently close to ω is a Λ -invariant symplectic form. Hence the classes $[\omega']$ fill out an open neighborhood of $[\omega]$ in $H^2(M)$, and so $B = \sigma_x - \sigma_y$. \square

Let J be any S^1 -invariant almost complex structure on M . The standard complex structure J_0 on \mathbb{C}^2 is also S^1 -invariant (under the diagonal action), and its restriction to S^3 preserves the contact planes $\ker \alpha$. Moreover, each vector $\xi \in T_p P_\Lambda$ can be considered as an equivalence class of vectors on $T(S^3 \times M)$; each such equivalence class has a unique representative in $\ker \alpha \oplus TM$ at each point in the S^1 -orbit $\text{pr}^{-1}(p)$. Therefore, the product complex structure $J_0 \times J$ on $\ker \alpha \oplus TM$ descends to an almost complex structure \tilde{J} on P_Λ . By construction, if J is compatible with ω , then \tilde{J} is compatible with Ω_c for all $c > \max K$. Moreover, \tilde{J} preserves the tangent spaces to the fibers, and the section σ_x is holomorphic for all fixed x .

Definition 2.3. *We define $\mathcal{J}_S(M)$ to be the set of all S^1 -invariant ω -compatible almost complex structures on M , and denote by $\mathcal{J}_S(P)$ the space of almost complex structures on P constructed as above from the elements $J \in \mathcal{J}_S(M)$. Note that $\mathcal{J}_S(P) \subset \mathcal{J}(P, \pi, \Omega_c)$ for all $c > \max K$.*

2.2. Small quantum homology. We shall work with quantum homology with coefficients in the ring $\Lambda := \Lambda^{\text{univ}}[q, q^{-1}]$ where q is a variable of degree 2 and Λ^{univ} is a generalized Laurent series ring in a variable t of degree 0:

$$\Lambda^{\text{univ}} := \left\{ \sum_{\kappa \in \mathbb{R}} r_\kappa t^\kappa \mid r_\kappa \in \mathbb{Q}, \#\{\kappa > c \mid r_\kappa \neq 0\} < \infty, \forall c \in \mathbb{R} \right\}.$$

Correspondingly, quantum cohomology has coefficients in the dual ring

$$\check{\Lambda} := \check{\Lambda}^{\text{univ}}[q, q^{-1}]$$

where q is as before and

$$\check{\Lambda}^{\text{univ}} := \left\{ \sum_{\kappa \in \mathbb{R}} r_\kappa t^\kappa \mid r_\kappa \in \mathbb{Q}, \#\{\kappa < c \mid r_\kappa \neq 0\} < \infty, \forall c \in \mathbb{R} \right\}.$$

Thus we define

$$\mathrm{QH}_*(M; \Lambda) = H_*(M, \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda, \quad \mathrm{QH}^*(M; \check{\Lambda}) = H^*(M, \mathbb{Q}) \otimes_{\mathbb{Q}} \check{\Lambda}.$$

These rings are \mathbb{Z} -graded in the obvious way:

$$(7) \quad \deg(a \otimes q^d t^\kappa) = \deg(a) + 2d,$$

where $a \in H_*(M)$ or $H^*(M)$. They also have $\mathbb{Z}/2\mathbb{Z}$ -gradings in which the even part is strictly commutative; for example,

$$\mathrm{QH}_{\mathrm{ev}} := H_{\mathrm{ev}}(M) \otimes \Lambda, \quad \mathrm{QH}_{\mathrm{odd}} := H_{\mathrm{odd}}(M) \otimes \Lambda.$$

Recall that the **quantum intersection product**

$$a * b \in \mathrm{QH}_{i+j-\dim M}(M; \Lambda), \quad \text{for } a \in H_i(M), b \in H_j(M)$$

is defined as follows:

$$a * b = \sum_{B \in H_2^S(M; \mathbb{Z})} (a * b)_B \otimes q^{-c_1(B)} t^{-\omega(B)},$$

where $(a * b)_B \in H_{i+j-\dim M+2c_1(B)}(M)$ is defined by the requirement that

$$(a * b)_B \cdot_M c = \mathrm{GW}_{B,3}^M(a, b, c) \quad \text{for all } c \in H_*(M).$$

Here $\mathrm{GW}_{B,3}^M(a, b, c) \in \mathbb{Q}$ denotes the Gromov–Witten invariant that counts the number of spheres in M in the class B that meet cycles representing the classes $a, b, c \in H_*(M)$. The product $*$ is extended to $\mathrm{QH}_*(M)$ by linearity over Λ , and is associative. Moreover, it respects the \mathbb{Z} -grading.

This product $*$ gives $\mathrm{QH}_*(M; \Lambda)$ the structure of a graded commutative ring with unit $\mathbb{1} = [M]$. Further, the invertible elements in $\mathrm{QH}_{\mathrm{ev}}(M; \Lambda)$ form a commutative group $\mathrm{QH}_{\mathrm{ev}}(M; \Lambda)^\times$ that acts on $\mathrm{QH}_*(M; \Lambda)$ by quantum multiplication.

We shall work mostly with quantum homology since this is more geometric. However, some examples mention quantum cohomology. The multiplication (quantum cup product) is defined via Poincaré duality: given $\alpha, \beta \in H^*(M)$ with Poincaré duals $a = \mathrm{PD}(\alpha), b = \mathrm{PD}(\beta)$

$$\alpha * \beta = \mathrm{PD}(a * b) = \sum_{B \in H_2^S(M; \mathbb{Z})} \mathrm{PD}((a * b)_B) \otimes q^{c_1(B)} t^{\omega(B)}.$$

Note that the coefficient is $q^{c_1(B)} t^{\omega(B)}$ rather than $q^{-c_1(B)} t^{-\omega(B)}$: in general the Poincaré duality map $\mathrm{PD} : \mathrm{QH}^*(M) \rightarrow \mathrm{QH}_*(M)$ is given by $\mathrm{PD}(\alpha \otimes q^d t^\kappa) = \mathrm{PD}(\alpha) \otimes q^{-d} t^{-\kappa}$. Thus in cohomology we must use the dual \check{v} of the valuation v , namely

$$(8) \quad \check{v}\left(\sum_{d, \kappa} a_{d, \kappa} \otimes q^d t^\kappa\right) = \min\{\kappa \mid \exists d : a_{d, \kappa} \neq 0\}.$$

2.3. The Seidel representation. In this paper, we will study Hamiltonian loops Λ by examining the geometry of the symplectic bundle P_Λ . On the classical level, this can be done by examining the (generalized) Weinstein homomorphism, which we mentioned in Lemma 1.8.

Proof of Lemma 1.8. Define the map

$$\mathcal{A}_{\omega, c} : \pi_1(\mathrm{Ham}(M, \omega)) \rightarrow \mathbb{R} \oplus \mathbb{Z}/\mathrm{im}(I_\omega \oplus I_c)$$

by

$$\mathcal{A}_{\omega, c}(\Lambda) = -[u_\Lambda(\sigma), c_{\mathrm{vert}}(\sigma)],$$

where $\sigma : S^2 \rightarrow P_\Lambda$ is any section. Note that this map is well defined. From the construction of the classes u_Λ and c_{vert} , and from the fact that $P_{\Lambda_1+\Lambda_2}$ is the fiber sum $P_{\Lambda_1} \#_M P_{\Lambda_2}$, it is easy to see that $\mathcal{A}_{\omega,c}$ is a homomorphism: see [9, Lemma 3.E]. Finally, if Λ is a circle with moment map K , then Lemma 2.2 implies that $u_\Lambda(\sigma_x) = -K(x)$ and $c_{\text{vert}}(\sigma_x) = m(x)$ for each fixed point x . \square

Definition 2.4. We define the **Seidel element** $\mathcal{S}(\Lambda) \in \text{QH}_{\dim M}(M; \Lambda)$ by

$$(9) \quad \mathcal{S}(\Lambda) = \sum_{\sigma \in H_2^{\text{sec}}(P)} a_\sigma \otimes q^{-c_{\text{vert}}(\sigma)} t^{-u_\Lambda(\sigma)}$$

where $a_\sigma \cdot_M c = \text{GW}_{\sigma,1}^{P_\Lambda}(c)$ for all $c \in H_*(M)$.⁴ Here $H_2^{\text{sec}}(P)$ denotes the affine subspace of $H_2(P; \mathbb{Z})$ that is represented by sections.

Intuitively, a_σ is represented by the class

$$\text{ev}_*(\mathcal{M}_{0,1}(P_\Lambda, \tilde{J}; \sigma)) \cap [M]$$

where $\mathcal{M}_{0,1}(P_\Lambda, \tilde{J}; \sigma)$ is the moduli space of all \tilde{J} -holomorphic sections in class σ with one marked point, ev is the obvious evaluation map to P_Λ and $[M]$ denotes the homology class represented by a fiber; see [13]. This moduli space has formal dimension $\dim M + 2c_{\text{vert}}(\sigma) + 2$. We find that $a_\sigma = 0$ unless

$$\deg(a_\sigma \otimes q^{-c_{\text{vert}}(\sigma)}) = \deg(a_\sigma) - 2c_{\text{vert}}(\sigma) = \dim M.$$

Because all dimensions are even, $\mathcal{S}(\Lambda)$ belongs to the strictly commutative part QH_{ev} of $\text{QH}_*(M)$. Moreover, $\mathcal{S}(\Lambda)$ is independent of the choice of symplectic extension form Ω since all of these are deformation equivalent. It is shown in [13] (using ideas from [23, 9]) that $\mathcal{S}(\Lambda)$ lies in $\text{QH}_{\text{ev}}(M; \Lambda)^\times$, the group of multiplicative units in the ring $\text{QH}_{\text{ev}}(M)$, and that the correspondence \mathcal{S} induces a group homomorphism

$$\mathcal{S} : \pi_1(\text{Ham}(M, \omega)) \longrightarrow \text{QH}_{\text{ev}}(M; \Lambda)^\times.$$

It is immediate from the definition that it lifts the Weinstein homomorphism.

It is often useful to identify $\text{QH}_{\text{ev}}(M; \Lambda)^\times$ with $\text{Aut}(\text{QH}_*(M; \Lambda))$, the group of automorphisms of $\text{QH}_*(M; \Lambda)$ as a right $\text{QH}_*(M; \Lambda)$ -module, since every such automorphism is determined by its value at $\mathbb{1}$. Correspondingly we define

$$\mathcal{S}(\Lambda)(a) := \mathcal{S}(\Lambda) * a \quad \forall a \in \text{QH}_*(M; \Lambda).$$

Since the Seidel element has degree $\dim M$, this endomorphism preserves degree.

Definition 2.5. The **Seidel representation** is the group homomorphism

$$\mathcal{S} : \pi_1(\text{Ham}) \longrightarrow \text{QH}_{\text{ev}}(M; \Lambda)^\times = \text{Aut}(\text{QH}_*(M; \Lambda)).$$

It is shown in [13] (see also [23, 9, 15]) that

$$(10) \quad \mathcal{S}(\Lambda)(a) = \sum_{\sigma \in H_2^{\text{sec}}(P)} b_\sigma \otimes q^{-c_{\text{vert}}(\sigma)} t^{-u_\Lambda(\sigma)}$$

⁴We use a 1-point Gromov–Witten invariant here. Similarly, we define $\mathcal{S}(\Lambda)(a)$ using a 2-point invariant. Because $[M] \cdot [\sigma] = 1$ for any section class, the divisor axiom for GW invariants implies that the 1-point invariant $\text{GW}_{\sigma,1}^P(c)$ equals the more usual 3-point invariant $\text{GW}_{\sigma,3}^P([M], [M], c)$. However, it is sometimes more convenient to use the 1-point invariant because the moduli space $\mathcal{M}_{0,1}$ can be compact while $\mathcal{M}_{0,2}$ never is because the two marked points must always be distinct.

where $b_\sigma \cdot_M c = \text{GW}_{\sigma,2}^{P_\Lambda}(a, c)$ for all $c \in \text{QH}_*(M)$. Here one should think of a as represented by a cycle in the fiber M_0 over the center of the disc D_0 , and b_σ and c as represented by cycles in the fiber M_∞ over the center of D_∞ . Then the element $\mathcal{S}(\Lambda)$ induces a ring isomorphism from $\text{QH}_*(M_0)$ to $\text{QH}_*(M_\infty)$. Intuitively, the class $\mathcal{S}(\Lambda)(a)$ is represented by the intersection of M_∞ and the space of all J -holomorphic sections of P_Λ that meet the cycle in M_0 which represents a . Since the connection in the bundle $(P, \Omega) \longrightarrow S^2$ provides an identification of M_0 with M_∞ that is well defined up to symplectic isotopy, $\mathcal{S}(\Lambda)$ gives rise to a well defined element of $\text{Aut}(\text{QH}_*(M; \Lambda))$ as claimed.

Example 2.6. Consider the rotation of the unit sphere S^2 with $K = 2\pi x_3$. Then the fibration P_Λ can be identified with the nontrivial fibration from the one point blow up M_* of \mathbb{CP}^2 to S^2 . By Lemma 2.2 the section σ_{\max} corresponding to the maximum (the north pole) has normal bundle of Chern number $m(n) = -1$, and so is the exceptional divisor, while the section σ_{\min} corresponding to the minimum (the south pole) has Chern number 1, and so lies in the class of a line. Since the Seidel element $\mathcal{S}(\Lambda_K)$ has degree $\dim M = 4$, a section σ can only contribute to it if $0 \geq 2c_{\text{vert}}(\sigma) = 2c_1(X) \geq -4$. Therefore, σ_{\max} is the only holomorphic section of P_Λ that can contribute to the Seidel element. It follows easily that $\mathcal{S}(\Lambda_K) = [pt] \otimes q t^{\omega(A)/2}$, as claimed in Example 1.10. \square

3. COMPUTING THE SEIDEL ELEMENT

This section contains the main proofs. We begin by calculating the contribution to the Seidel element $\mathcal{S}(\Lambda)$ of the sections σ_{\max} through points on the maximal fixed set F_{\max} . In Proposition 3.3 we show that this is nonzero precisely when F_{\max} is homologically visible. These arguments use easy results on the behavior of J -holomorphic spheres. To go further, we need a version of the localization theorem: there is a T^2 -action on the moduli spaces of stable maps and only the invariant elements contribute to $\mathcal{S}(\Lambda)$. This theorem is stated in §3.2. We defer the proof to §4.2, devoting the rest of this section to an investigation of the invariant elements. We first prove Theorem 1.9. Then, in §3.3, we consider the semifree case, and prove Theorem 1.14. Finally, in §3.4, we consider the case where the isotropy is at most twofold, and prove Theorems 1.16 and 1.17.

3.1. The contribution of the maximal fixed set. We begin with some preliminary remarks about \tilde{J} -holomorphic sections of $P := P_\Lambda$. Throughout we assume $\tilde{J} \in \mathcal{J}_S(P_\Lambda)$, the space of almost complex structures on P_Λ that are constructed from S^1 -invariant almost complex structures on M using the identification of P_Λ with a quotient of $S^3 \times M$. See Definition 2.3.

Let $\mathcal{M}_{0,k}(P, \tilde{J}; \sigma)$ denote the space of equivalence classes $[u, \mathbf{z}]$ of \tilde{J} -holomorphic maps $u : S^2 \longrightarrow P$ in class A with k pairwise distinct marked points $\mathbf{z} := \{z_1, \dots, z_k\}$. Here, two such pairs (u, \mathbf{z}) and (u', \mathbf{z}') are equivalent if there is $\psi \in \text{PSL}(2, \mathbb{C})$ such that

$$u' = u \circ \psi, \quad \psi(z'_i) = z_i, \quad i = 1, \dots, k.$$

The compactification $\overline{\mathcal{M}}_{0,k}(P, \tilde{J}; \sigma)$ consists of equivalence classes $\tau = [\Sigma(\mathbf{u}), \mathbf{u}, \mathbf{z}]$ of \tilde{J} -holomorphic stable maps $\mathbf{u} : \Sigma(\mathbf{u}) \longrightarrow P$ with k marked points. Here $\Sigma(\mathbf{u})$ is a union of copies of S^2 attached via a tree graph, and the equivalence relation is given by all reparametrizations that respect the special points, i.e. the attaching

(or nodal) points and the marked points. If σ is a section class, each element τ in $\overline{\mathcal{M}}_{0,k}(P, \tilde{J}; \sigma)$ projects via $\pi : P \rightarrow S^2$ to an equivalence class of holomorphic maps $\pi \circ \mathbf{u} : \Sigma(u) \rightarrow S^2$ of total degree 1. Such a map has just one component of degree 1; on all the other components $\pi \circ \mathbf{u}$ is constant. Thus τ has a distinguished component that is a section, called the **root**. The other components are mapped into fibers.

We shall be specially interested in the case when $k = 2$ and the first marked point is mapped to M_0 , the other to M_∞ . In this case, there is a unique chain of spheres joining the component that contains the first marked point z_0 to the component that contains the second marked point z_∞ ; we call the components of this chain the **principle components**. The other spheres are called **bubbles**. The root is always a principle component.

For most of the results in this paper, we will need to look at invariant chains, as described in the next subsection. However, the following observation, due to Seidel⁵, allows us to give a simpler argument when we are studying curves in a class $\sigma_{\max} + B$ with $\omega(B) \leq 0$.

Lemma 3.1. *Consider a Hamiltonian circle action Λ_K on a compact symplectic manifold (M, ω) , and let $\tilde{J} \in \mathcal{J}_S(P_\Lambda)$ be constructed from $J \in \mathcal{J}_S(M)$. Fix $B \in H_2^S(M; \mathbb{Z})$, and consider the moduli space*

$$\overline{\mathcal{M}}_{0,0}(P_\Lambda, \tilde{J}; \sigma_{\max} + B).$$

- If $B \neq 0$ and $\omega(B) \leq 0$, the moduli space is empty.
- If $B = 0$, the moduli space is compact and can be identified with F_{\max} itself.

Proof. The symplectic form Ω_c defined in (5) is compatible with \tilde{J} for any $c > \max K$. Fix $[z, w] \in P_\Lambda = S^3 \times_{S^1} M$. Recall that any non-zero tangent vector $\xi \in T_{[z,x]}P_\Lambda$ can be uniquely represented by a vector $h + v \in T_{(z,x)}(S^3 \times M)$, where $h \in \ker \alpha \subset T_z S^3$ and $v \in T_x M$. Now

$$\begin{aligned} \Omega_c(\xi, \tilde{J}\xi) &= (\omega - dK \wedge \alpha + (c - K)d\alpha)(h + v, J_0 h + Jv) \\ &= \omega(v, Jv) + (c - K)d\alpha(h, J_0 h) \\ &\geq (c - K_{\max})d\alpha(h, J_0 h) \\ &= (c - K_{\max})\chi^* \tau(\xi, \tilde{J}\xi) \end{aligned}$$

with equality impossible unless $v = 0$ and $K(x) = \max K$. Since $\chi^* \tau$ is the pullback by the Hopf map of the area form on S^2 with area 1, it follows that for any \tilde{J} -holomorphic section σ

$$\Omega_c(\sigma) \geq c - K_{\max} = \Omega_c(\sigma_{\max}),$$

with equality occurring exactly if σ is a constant section σ_x for some $x \in F_{\max}$.

Since every stable map in a section class σ either consists of a section, or is the union of a section with other spheres A_i which lie in the fibers and satisfy $\omega(A_i) > 0$, the only stable maps that represent a section class σ with $\Omega_c(\sigma) \leq c - \max K$ are the constant sections $\sigma_x, x \in F_{\max}$. The result follows. \square

Lemma 3.2. *Let x be any fixed point of the S^1 -action. For each $\tilde{J} \in \mathcal{J}_S(P_\Lambda)$, the \tilde{J} -holomorphic curve σ_x is regular precisely when the negative weights at x are all equal to -1 .*

⁵Private communication.

Proof. Recall that σ_x is regular if and only if the linearization D_u of the corresponding Cauchy–Riemann operator is surjective. When $J \in \mathcal{J}_S(P_\Lambda)$ the normal bundle of σ_x is holomorphic and splits into a sum of line bundles $\oplus_i L_i$ that are preserved by D_u . Moreover, D_u restricts on each L_i to the usual Dolbeault delbar operator. Thus D_u is surjective precisely when $c_1(L_i) \geq -1$ for all i . \square

The next proposition generalizes part (i) of Theorem 1.9.

Proposition 3.3. *Consider a Hamiltonian circle action Λ_K on a compact symplectic manifold (M, ω) with normalized moment map K . Let $e(\mathcal{E}_{\max}) \in H_*(F_{\max})$ denote the Poincaré dual of the Euler class of the obstruction bundle at F_{\max} (see equation (1)), denote the inclusion $H_*(F_{\max}) \rightarrow H_*(M)$ by ι , and set $K_{\max} := K(F_{\max})$ and $m_{\max} := m(F_{\max})$. Then:*

$$\mathcal{S}(\Lambda_K) = \iota(e(\mathcal{E}_{\max})) \otimes q^{-m_{\max}} t^{K_{\max}} + \sum_{B \in H_2^S(M; \omega(B) > 0)} a_B \otimes q^{-m_{\max} - c_1(B)} t^{K_{\max} - \omega(B)}.$$

Proof. By Lemma 2.2, $u_\Lambda(\sigma_{\max}) = -K_{\max}$ and $c_{\text{vert}}(\sigma_{\max}) = m_{\max}$. Therefore we may write

$$\mathcal{S}(\Lambda_K) = \sum_{B \in H_2^S(M; \mathbb{Z})} a_B \otimes q^{-m_{\max} - c_1(B)} t^{K_{\max} - \omega(B)},$$

where a_B is the contribution from the section class $\sigma_{\max} + B$. Fix any $c \in H_*(M)$. It is enough to show that $a_0 \cdot_M c = \iota(e(\mathcal{E}_{\max})) \cdot_M c$, and that $a_B \cdot_M c = 0$ for every nonzero $B \in H_2^S(M; \mathbb{Z})$ such that $\omega(B) \leq 0$. By definition, $a_B \cdot_M c = \text{GW}_{\sigma_{\max} + B, 1}^{P_\Lambda}(c)$. Choose an almost complex structure $\tilde{J} \in \mathcal{J}_S(P)$. By Lemma 3.1, if $\omega(B) \leq 0$ and $B \neq 0$ then the moduli space $\mathcal{M}_{0,0}(P_\Lambda, \tilde{J}, \sigma_{\max} + B)$ is empty, and so $a_B \cdot_M c = 0$, as required. On the other hand, $\mathcal{M}_{0,1}(P_\Lambda, \tilde{J}, \sigma_{\max})$ can be identified with the compact manifold $S^2 \times F_{\max}$. (The S^2 -factor is the locus of the single marked point.)

If F_{\max} is semifree, then Lemma 3.2 implies that σ_x is regular for every $x \in F_{\max}$. Hence the intersection of the evaluation pseudocycle $\text{ev} : \mathcal{M}_{0,1}(P, \tilde{J}, \sigma_{\max}) \rightarrow M_\infty$ with any class c in the fiber M_∞ is precisely $[F_{\max}] \cdot c$. Thus $a_0 = [F_{\max}]$ in this case.

If any of the negative weights $-k_i$ at F_{\max} is less than -1 , the elements of the compact manifold $\mathcal{M} := \mathcal{M}_{0,1}(P, \tilde{J}, \sigma_{\max})$ are not regular. Rather, for each i , the cokernel of the restriction $D_{u_x} : C^\infty(S^2, E_i) \rightarrow \Omega^{0,1}(S^2, E_i)$ is a vector space of dimension $\dim E_i \otimes \mathbb{C}^{k_i-1}$, and as x varies in F_{\max} these cokernels fit together to form the bundle $E_i \otimes \mathbb{C}^{k_i-1}$ over \mathcal{M} . Thus the total obstruction bundle is the bundle $\mathcal{E} \rightarrow \mathcal{M}$ of equation (1). It follows from the standard theory (see for example [12, §5.3] or [15, Chapter 7.2]) that the regularized moduli space corresponds to the zero set of a generic section of $\mathcal{E} =: \mathcal{E}_{\max}$. Therefore $\text{GW}_{\sigma_{\max}, 1}^{P_\Lambda}(c) = \iota(e(\mathcal{E})) \cdot_M c$ for each $c \in H_*(M)$, and the result follows. \square

3.2. Invariant beads and chains. In order to understand the moduli spaces of sections in an arbitrary class σ we exploit the fact that T^2 acts on $(P_\Lambda, \Omega, \tilde{J})$ when $\tilde{J} \in \mathcal{J}_S(P_\Lambda)$. Here the first factor $S^1 \times \{1\}$ acts on P_Λ by rotating the fibers via ϕ_t while the second factor $\{1\} \times S^1$ acts by rotating the base as follows:

$$\theta \cdot [z, 1; x] = [e^{2\pi i \theta} z, 1; \phi_t x], \quad \theta \cdot [1, z; x] = [1, e^{-2\pi i \theta} z; x].$$

Note that the only points of P_Λ fixed by the whole group are the points in M_0 and M_∞ that are fixed by the original S^1 -action ϕ_t . Because the elements of $\mathcal{J}_S(P_\Lambda)$ are constructed from S^1 -invariant almost complex structures on M (see Definition 2.3), this action preserves \tilde{J} . Hence T^2 acts on the moduli spaces of stable maps via postcomposition.

The next result is a version of the localization principle for T^2 -actions; it is well known in the algebraic case and is proved in the symplectic situation in §4.2. Given two (weighted) pseudocycles $f : Z \rightarrow P_\Lambda$ and $f' : Z' \rightarrow P_\Lambda$ (see Definition 4.1) and a section class σ , we define

$$\overline{\mathcal{M}}_{0,2}(P_\Lambda, \tilde{J}, \sigma; Z, Z') := \text{ev}^{-1}(\overline{f(Z)} \times \overline{f'(Z')})$$

where $\text{ev} : \overline{\mathcal{M}}_{0,2}(P_\Lambda, \tilde{J}, \sigma) \rightarrow P_\Lambda \times P_\Lambda$ is the evaluation map. The pseudocycles are said to be **S^1 -invariant** if the images $f(Z)$ and $f'(Z')$ are closed under the action of S^1 . In this case, if $\tilde{J} \in \mathcal{J}_S(P_\Lambda)$, then clearly there is an induced action of T^2 on this cutdown moduli space.

Proposition 3.4. *Suppose that $f : Z \rightarrow M_0$ and $f' : Z' \rightarrow M_\infty$ are S^1 -invariant weighted pseudocycles which represent the classes a and a' in $H_*(M)$, respectively. Given $\tilde{J} \in \mathcal{J}_S(P_\Lambda)$, write*

$$\mathcal{S}(\Lambda)(a) = \sum_{\sigma \in H_2^{\text{sec}}(P)} a_\sigma \otimes q^{-c_{\text{vert}}(\sigma)} t^{-u_\Lambda(\sigma)}.$$

Then $a_\sigma \cdot_M a' = 0$ unless the moduli space $\overline{\mathcal{M}}^{\text{cut}} := \overline{\mathcal{M}}_{0,2}(P_\Lambda, \tilde{J}, \sigma; Z, Z')$ contains a T^2 -invariant element. Moreover, $a_\sigma \cdot_M a'$ is a sum of contributions, one from each connected component of the space $(\overline{\mathcal{M}}^{\text{cut}})^{T^2}$ of invariant elements.

Note that most T^2 -invariant elements in $\overline{\mathcal{M}}_{0,2}(P_\Lambda, \tilde{J}, \sigma; Z, Z')$ are not regular. Therefore it would be a nontrivial task to calculate their actual contributions to the invariant. In this paper we do not attempt such calculations.

The next task is to figure out the structure of the T^2 -invariant elements in $\overline{\mathcal{M}}_{0,2}(P_\Lambda, \tilde{J}, \sigma)$. Note that each principal component has 2 special points joining it to the other principal components. We will place these at 0 and ∞ and then identify $S^2 \setminus \{0, \infty\}$ with the cylinder $(s, t) \in \mathbb{R} \times S^1$ with complex structure j_0 defined by $j_0(\partial_s) = \partial_t$.

Lemma 3.5. *Let $\tilde{J} \in \mathcal{J}_S(P_\Lambda)$ be constructed from $J \in \mathcal{J}_S(M)$ and denote by g_J the metric on M defined by J and ω .*

(i) *If A is a section class the only elements in $\mathcal{M}_{0,2}(P_\Lambda, \tilde{J}, A)$ that are fixed by the T^2 -action have the form $[u; 0, \infty]$ where $u : S^2 \rightarrow P_\Lambda$ is parametrized as a section and has as image some constant sphere σ_x where $x \in M^{S^1}$.*

(ii) *If $A \in H_2(M)$ then the only elements in $\mathcal{M}_{0,k}(P_\Lambda, \tilde{J}, A)$ that are fixed by the T^2 -action lie in either M_0 or M_∞ . If such an element does not lie entirely in M^{S^1} , then $k \leq 2$ and there exists a parametrization $u : \mathbb{R} \times S^1 \rightarrow M$ and a path $\gamma : \mathbb{R} \rightarrow M$ which joins two fixed points x and y in M so that the marked points lie in $u^{-1}(\{x, y\})$, and*

$$(11) \quad u(s, t) = \phi_{pt/q} \gamma(s), \quad \text{and} \quad \gamma'(s) = \frac{p}{q} \text{grad}_{g_J} K,$$

where $p \neq 0$ and where $q > 0$ is the order of the isotropy group of the points in the image of γ . There is a unique choice of parametrization such that

$$(12) \quad \lim_{s \rightarrow -\infty} u(s, t) = x \quad \text{and} \quad \lim_{s \rightarrow \infty} u(s, t) = y.$$

Proof. Statement (i) is clear, as is the first claim in (ii). Thus, identifying M_0 and M_∞ with M , we just need to understand the spheres $u : S^2 \rightarrow M$ that are fixed by Λ and do not lie entirely in M^{S^1} . Suppose first that u is simple, i.e. not multiply covered. For each $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$, the composite $\phi_\theta \circ u$ must be a reparametrization of u , that is, there is a unique $\psi_\theta \in \text{PSL}(2, \mathbb{C})$ such that $\phi_\theta \circ u = u \circ \psi_\theta$. It follows easily the assignment $\theta \mapsto \psi_\theta$ defines a homomorphism $S^1 \rightarrow \text{PSL}(2, \mathbb{C})$. Since the only circle subgroups of $\text{PSL}(2, \mathbb{C})$ consist of rotations about a fixed axis, there are two points in S^2 , say $0, \infty$, that are mapped by u into M^{S^1} . If $\text{im } u \cap M^{S^1} = \{x, y\}$ we may suppose $u(0) = x$ and $u(\infty) = y$. If there are marked points on u they must form a subset of $\{0, \infty\}$. Using coordinates (s, t) on $S^2 \setminus \{0, \infty\} = \mathbb{R} \times S^1$ as above, we find that for some $q \neq 0$

$$\psi_\theta(s, t) = (s, t + q\theta), \quad \phi_\theta \circ u(s, t) = u(s, t + q\theta).$$

Thus, $\text{im } u$ lies in the set of points with isotropy group $\mathbb{Z}/(|q|)$. Denoting $\gamma(s) := u(s, 0)$, we have $u(s, \theta) = \phi_{\theta/q} \gamma(s)$. Moreover

$$0 = \partial_s u + J \partial_t u = (\phi_{t/q})_* (\gamma'(s) + \frac{1}{q} J X_K(\gamma(s))),$$

where X_K is the Hamiltonian flow induced by K . Thus $\gamma' = \frac{1}{q} \text{grad } K$ because $-JX_K = \text{grad } K$. (Here we take the gradient with respect to the metric g_J .) Since every sphere is the $|p|$ -fold cover of a simple sphere, this proves (ii). To get the stated result, we absorb any negative sign into p rather than q . \square

Definition 3.6. Let x and y be two fixed points in M .

For $q > 0$ and $p \neq 0$, a **bead from x to y of type (p, q)** is a map $u : \mathbb{R} \times S^1 \rightarrow M$ which satisfies equations (11) and (12).

For $q = 0$ and $p > 0$, a **bead from x to y of type (p, q)** is a p -fold cover of a simple J -holomorphic sphere $u : \mathbb{R} \times S^1 \rightarrow M$ that lies entirely in one component of the fixed point set M^{S^1} and which satisfies equation (12).

Definition 3.7. Given $x, y, z \in M^{S^1}$ an **invariant principal chain from x to y in class $\sigma_z + \mathbf{A}$ and with root z** is a sequence of critical points $x = x_1, x_2, \dots, x_k = y$ of K joined by invariant \tilde{J} -holomorphic spheres with the following properties:

- (a) there is $1 \leq i_0 \leq k$ such that $x_{i_0} = x_{i_0+1} = z$ and these points are joined by the section σ_z ;
- (b) for each $1 \leq i < k$ where $i \neq i_0$, the points x_i, x_{i+1} are joined by a (p_i, q_i) -bead in class A_i ;
- (c) $\sum_{i \neq i_0} A_i = A$.

Further an **invariant chain from x to y in class $\sigma_z + \mathbf{A}$ and with root z** is a chain as above with additional ghost components at each of which a T^2 -invariant tree of (p, q) beads is attached. In this case, A is the sum of the classes represented by the principal spheres and the bubbles.

The next lemma is an immediate consequence of Lemma 3.5 and the above definitions.

Lemma 3.8. *Let $f : Z \rightarrow P_\Lambda$ and $f' : Z' \rightarrow P_\Lambda$ be T^2 -invariant pseudocycles, σ be a section class and choose $\tilde{J} \in \mathcal{J}_S(P_\Lambda)$. Then every T^2 -invariant element of the cut down moduli space $\overline{\mathcal{M}}_{0,2}(P_\Lambda, \tilde{J}, \sigma; Z, Z')$ is an invariant chain from a point $x \in \overline{f(Z)}$ to a point $y \in \overline{f'(Z')}$.*

We will need the following useful facts about beads.

Lemma 3.9. *Choose $J \in \mathcal{J}_S(M)$ and consider a $(J$ -holomorphic) (p, q) -bead from x to y in class A . If $q \neq 0$, then $A = p(\sigma_x - \sigma_y)/q$. Further:*

- (i) *If $K(y) > K(x)$ then $p > 0$, $\omega(A) = p|K(x) - K(y)|/q$ and $c_1(A) = p(m(x) - m(y))/q$.*
- (ii) *If $K(y) < K(x)$ then $p < 0$, $\omega(A) = |p||K(x) - K(y)|/q$ and $c_1(A) = p(m(x) - m(y))/q$.*
- (iii) *If $K(y) = K(x)$ then $\omega(A) > |K(y) - K(x)|$.*

Proof. We saw in Lemma 2.2 that the homology class of the sphere formed by the Λ -orbit of an arc going from x to y is $\sigma_x - \sigma_y$. Hence each (p, q) bead from x to y lies in the class $A = p(\sigma_x - \sigma_y)/q$ where $\omega(A) = p(K(y) - K(x))/q$. Statements (i), (ii) and (iii) now follow from the fact that $\omega(A) > 0$ and $c_1(A) = p(m(x) - m(y))/q$. \square

The proofs of our other results are based on a more careful study of the structure of the T^2 -invariant elements in $\overline{\mathcal{M}}_{0,2}(P_\Lambda, \tilde{J}, \sigma_\Lambda + B)$. We begin by slightly strengthening the conclusion of Proposition 3.4.

Lemma 3.10. *Consider a Hamiltonian circle action Λ_K on a compact symplectic manifold (M, ω) with normalized moment map K . Let F_{\max} be the maximal fixed component and choose $J \in \mathcal{J}_S(M)$. Given $B \in H_2^S(M)$, let a_B denote the contribution of $\sigma_{\max} + B$ to the Seidel element $\mathcal{S}(\Lambda_K)$. Then $a_B = 0$ unless B can be represented by an invariant J -holomorphic stable map that intersects F_{\max} . More generally, if $f' : Z' \rightarrow M$ is an invariant pseudocycle representing the class a' , then $a_B \cdot_M a' = 0$ unless B can be represented by an invariant J -holomorphic stable map that intersects both F_{\max} and $\overline{f'(Z')}$.*

Proof. Assume $a_B \cdot_M a' \neq 0$. By Proposition 3.4, there must be a T^2 -invariant element in

$$\overline{\mathcal{M}}_{0,2}(P_\Lambda, \tilde{J}, \sigma_{\max} + B; M_0, \overline{Z'}),$$

where $\tilde{J} \in \mathcal{J}_S(P_\Lambda)$ is constructed from J in the usual way. Hence, by Lemma 3.8, there is an invariant chain from $x \in M_0$ to $y \in \overline{f'(Z')}$ in the class $\sigma_{\max} + B$. Let z denote its root. Let A' be the sum of the homology classes represented by the subchain of spheres in M_0 from x to z , and let A'' be the sum of the homology classes represented by the subchain of spheres in M_∞ from z to y . Then $A' + A'' + \sigma_z = \sigma_{\max} + B$. Since the orbit of an upward gradient flow line from z to F_{\max} is J -holomorphic, the class $\sigma_z - \sigma_{\max}$ is also represented by a J -holomorphic sphere. \square

Proof of Theorem 1.9. Part (i) is included in Proposition 3.3. Part (ii) follows immediately from Lemma 3.10.

Now assume that (M, J) is NEF and that $2c_1(B') \geq \text{codim } F_{\max}$ for all J -holomorphic spheres B' that do not lie entirely in F_{\max} . Assume also that $a_B \neq 0$. By Lemma 3.10, B can be represented by a J -holomorphic stable map which intersects F_{\max} . We must show that all components of this stable map lie in F_{\max} . Suppose the contrary. Then the assumptions imply that $2c_1(B) \geq \text{codim } F_{\max}$. On the other hand, $0 \leq \deg(a_B) = \dim F_{\max} + 2c_1(B) \leq \dim M$. Therefore, $2c_1(B) = \text{codim } F_{\max}$. Therefore, $\deg(a_B) = \dim M$. Since a_B is not zero, this implies that it is a multiple of the generator of $H_{\dim M}(M)$; hence $a_B \cdot [pt] \neq 0$. Choose $y \in F_{\min}$. Then since $a_B \cap [y] \neq 0$, Lemma 3.10 implies B can be represented by a J -holomorphic stable map which intersects F_{\max} and y . Let B_1 be a sphere in the corresponding stable map which intersects F_{\max} at x_1 but does not lie entirely in F_{\max} . Let x_2 denote the second marked point in B_1 . Let B_2, \dots, B_k be the remaining J -holomorphic spheres in B . Then

$$\text{codim } (F_{\max}) = 2c_1(B) = 2 \sum_{i=1}^k c_1(B_i).$$

Since the assumptions imply that $c_1(B_i) \geq 0$ for all i and $2c_1(B_1) \geq \text{codim } (F_{\max})$, we conclude that $2c_1(B_1) = \text{codim } (F_{\max})$ and $c_1(B_i) = 0$ for all $i \neq 1$. Since F_{\max} is semifree, B_1 is a bead of type (p, q) with $q = 1$. By Lemma 3.9(ii), $p < 0$ and

$$2c_1(B_1) = 2p(m(F_{\max}) - m(x_2)) = -2pm(x_2) - p \text{codim } (F_{\max}).$$

Since $2c_1(B_1) = \text{codim } (F_{\max})$, $m(x_2) \leq 0$. Since $c_1(B_i) = 0$ for all $i \neq 1$, the next bead on the principal chain must connect x_2 to another point, x_3 , which also satisfies $m(x_3) \leq 0$. Proceeding inductively, we see that $m(y) \leq 0$. But this is impossible, because $m(y) > 0$ for all $y \in F_{\min}$. \square

Proof of Proposition 1.6. Since $\mathcal{S}(\Lambda_K) = \mathbb{1}$, there is a class B with

$$\omega(B) = -u_\Lambda(\sigma_{\max}) = K_{\max} \quad \text{and} \quad c_1(B) = -c^{\text{vert}}(\sigma_{\max}) = -m_{\max},$$

such that $a_B \cdot_M [pt] \neq 0$. By Lemma 3.10 there is an invariant J -holomorphic stable map in class B that intersects F_{\max} and F_{\min} . Let B_1, B_2, \dots, B_j be the beads in the principal chain which do not lie in a single fixed component. Note that $\omega(B) \geq \sum_{i=1}^j \omega(B_i)$, with equality impossible unless $B = \sum B_i$.

Let the second marked point of the bead B_i lie in the fixed component F_i . Since B_i does not lie in a single fixed component, $K(F_i) \neq K(F_{i-1})$. Obviously, F_{i-1} and F_i cannot be joined by a bead of type (p, q) , where $q > 0$, unless they lie in the same component of $M^{\mathbb{Z}/(q)}$. Hence, by Lemma 3.9, $\omega(B_i) \geq \frac{|K(F_{i-1}) - K(F_i)|}{q(F_{i-1}, F_i)}$. If $K(F_{i-1}) - K(F_i) > 0$, then equality is impossible unless $p = -1$, in which case $-c_1(B_i) = \frac{m(F_{i-1}) - m(F_i)}{q(F_{i-1}, F_i)}$. If $K(F_{i-1}) - K(F_i) < 0$, then equality is impossible unless $p = 1$, in which case $-c_1(B_i) = -\frac{m(F_{i-1}) - m(F_i)}{q(F_{i-1}, F_i)}$. Thus,

$$\sum_{i=1}^j \omega(B_i) \geq \sum_{i=1}^j \frac{|K(F_{i-1}) - K(F_i)|}{q(F_{i-1}, F_i)},$$

with equality impossible unless

$$m_{\max} = -\sum_{i=1}^j c_1(B_i) = \sum_{i=1}^j \frac{m(F_{i-1}) - m(F_i)}{q(F_{i-1}, F_i)} \cdot \frac{K(F_{i-1}) - K(F_i)}{|K(F_{i-1}) - K(F_i)|}.$$

\square

One can formulate analogous results for the intermediate fixed components F . Consider the function $R : \mathcal{C} \longrightarrow \mathbb{R}$, where $\mathcal{C} \subset \mathbb{R}$ is the set of critical values of K . For $\mu \in \mathcal{C}$ we define $R(\mu)$ to be the infimum of the set of μ' such that $F^\mu H_*(M) \subset F_{\mu'} H_*(M)$. (For notation, see the discussion after Remark 1.11.) In other words, every class c^- , where $c \in H_*(F)$ for some F with $K(F) \leq \mu$, is a linear combination of classes $(c')^+$ where $c' \in H_*(F')$ for some F' with $K(F') \geq R(\mu)$.

Proposition 3.11. *Suppose that Λ_K is inessential. Then for every critical value $\mu \in \mathcal{C}$, there is an invariant chain in some class $\sigma_z + B$ with $u_\Lambda(\sigma_z + B) = c^{\text{vert}}(\sigma_z + B) = 0$ from a critical point x with $K(x) \leq \mu$ to another critical point y with $K(y) \leq R(\mu)$.*

Proof. If $c^- \in F^\mu H_*(M)$ then $\mathcal{S}(\Lambda_K)(c^-) = c^-$ is represented in $F_{R(\mu)} H_*(M)$. Hence $\mathcal{S}(\Lambda_K)(c^-) \cdot_M c' \neq 0$ for some $c' \in F^{R(\mu)} H_*(M)$. By Proposition 3.4 this is possible only if there is a chain in class $\sigma_z + B$ with the given properties. \square

3.3. The semifree case. Suppose that the moment map K generates a semifree S^1 -action. By Lemma 4.5, for a generic almost complex structure $J \in \mathcal{J}_S(M)$, the pair (K, g_J) is Morse regular where g_J denotes the metric associated to J (see Definition 4.2). Let F and F' be fixed components. By Lemma 4.6, for any generic submanifolds C of F and C' of F' , the unstable manifolds $W^u(C)$ and $W^u(C')$ are pseudocycles. We shall denote them by $W_J^u(C)$ and $W_J^u(C')$ to emphasize that they depend on the choice of J . By construction, these unstable manifolds are S^1 -invariant. Hence, to prove Theorem 1.14, we only need analyze the invariant chains in the moduli space $\overline{\mathcal{M}}_{0,2}(P_\Lambda, \tilde{J}, \sigma_F + B; W_J^u(C), W_J^u(C'))$, where $\omega(B) \leq 0$.

Lemma 3.12. *Consider a semifree Hamiltonian circle action Λ_K on a compact symplectic manifold (M, ω) . Let \tilde{J} be a generic almost complex structure in $\mathcal{J}_S(P)$. Let F and F' be connected components of the fixed point set and let $C \subset F$ and $C' \subset F'$ be generic submanifolds. Fix $B \in H_2^S(M)$ such that $\omega(B) \leq 0$, and consider the moduli space*

$$\overline{\mathcal{M}}_{0,2}(P_\Lambda, \tilde{J}, \sigma_F + B; W_J^u(C), W_J^u(C')).$$

- (i) *If $B \neq 0$, the moduli space contains no invariant chains.*
- (ii) *If $F \neq F'$, there are no invariant chains unless $\dim(W_J^u(C)) + \dim(W_J^u(C')) > \dim M$.*
- (iii) *If $B = 0$ and $F = F'$, the only invariant chains are the constant sections σ_x for $x \in C \cap C'$.*

Proof. Assume that there is an invariant chain from $x \in \overline{W_J^u(C)}$ to $y \in \overline{W_J^u(C')}$ with root z in the class $\sigma_F + B$. Note immediately that

$$K(x) \leq K(F),$$

with equality if and only if $x \in C$. Let A' and A'' denote the classes represented by the invariant subchains from x to z , and from z to x , respectively. Then $A' + A'' + \sigma_z = \sigma_F + B$, and so by Lemma 2.2

$$\omega(A') + \omega(A'') - K(z) + K(F) = \omega(B) \leq 0.$$

Because the action is semifree, every bead of type (p, q) in the invariant chain from x to z has $q = 1$. Hence, by Lemma 3.9

$$K(z) - K(x) \leq \omega(A')$$

with equality impossible unless $K(x) \leq K(z)$, A' is the class of a chain of $(1, 1)$ beads from x to z , and $A' = \sigma_x - \sigma_z$. This implies both that there is a broken K -trajectory from z to x , and that

$$0 \leq \omega(A''),$$

with equality if and only if $A'' = 0$. In this case, $z = y \in \overline{W_J^u(C')}$. Therefore, $K(z) \leq K(F')$.

Considering all four displayed inequalities together, it is clear that in fact they must all be equalities. This implies that $A' = \sigma_x - \sigma_z$ and $A'' = 0$, and also that $x \in C \subset F$, so $\sigma_x = \sigma_F$. Therefore $B = A' + A'' + \sigma_z - \sigma_F = 0$. This proves (i).

Next, since it implies both that $x \in C$ and that there is a broken K -trajectory from z to x , there is a broken K -trajectory from z to C . Additionally, since $z \in \overline{W_J^u(C')}$, by Lemma 4.4, there is a broken K -trajectory from C' to z . Therefore, there is a broken K -trajectory from C' to C . If $F \neq F'$, then by Lemma 4.3, this implies that $\dim W_J^s(C) + \dim W_J^u(C') > \dim M$. This proves (ii).

Finally, assume that $F = F'$. Then since $K(x) = K(F)$, $K(x) \leq K(z)$, and $K(z) \leq K(F')$, it follows that $K(x) = K(z)$. Thus $z \in F$ and $A' = \sigma_F - \sigma_z = 0$. Since also $A'' = 0$, the last claim follows. \square

Before proving the rest of the theorems from the first section, we need to consider the contributions of fixed point sets other than F_{\max} . To simplify the proof of Lemma 3.14 below, it is convenient to work with almost complex structures on M that are well behaved near the fixed components. Each fixed component F has a neighborhood \mathcal{N}_F that can be identified with a neighborhood of the zero section in a sum of Hermitian vector bundles $\pi_F : E_1 \oplus \cdots \oplus E_k \longrightarrow F$ in such a way that the moment map K is given by

$$(13) \quad K(v_1, \dots, v_k) = \sum_j \pi m_j \|v_j\|^2, \quad m_j \in \mathbb{Z} \setminus \{0\}$$

and S^1 acts in E_j by rotation by $e^{2\pi i m_j}$. The symplectic connection with horizontal spaces Hor_x equal to the ω -orthogonals to the fibers is also S^1 -invariant. Therefore, starting from any ω -compatible J_F on the components F , we may extend J_F to an S^1 -invariant ω -compatible almost complex structure J_M on M whose restriction to each set \mathcal{N}_F agrees with the complex structure on the fibers of π_F , leaves the horizontal distribution invariant and is such that π_F is holomorphic.

Definition 3.13. Fix once and for all such an almost complex structure J_M on M . Define $\mathcal{J}_S^n(M)$ to be the set of all S^1 -invariant ω -compatible almost complex structures on M that equal J_M near the fixed point components F . Let $\mathcal{J}_S^n(P_\Lambda)$ denote the subspace of $\mathcal{J}_S(P_\Lambda)$ constructed from $J \in \mathcal{J}_S^n(M)$ as in Definition 3.13.

Thus when $J \in \mathcal{J}_S^n(M)$ each fixed point component F has a neighborhood \mathcal{N}_F that can be identified with a neighborhood of the zero section in the complex vector bundle $\pi_F : E^+ \oplus E^- \longrightarrow F$, where E^+ (resp. E^-) is the subbundle of the normal bundle with positive (resp. negative) weights. Moreover, π_F is J -holomorphic. Hence a neighborhood of the submanifold $S^2 \times F$ in P_Λ can be identified with a neighborhood of the zero section in

$$\tilde{\pi}_F : \tilde{E}^+ \oplus \tilde{E}^- \longrightarrow S^2 \times F,$$

where the bundle $\tilde{E}^\pm \longrightarrow S^2 \times F$ is induced in the obvious way from the S^1 -action. Moreover, $\tilde{\pi}_F$ is \tilde{J} -holomorphic. Denote by $\mathcal{M}_{0,2}^{\text{sec}}(P_\Lambda, \tilde{J}, \sigma_F)$ the moduli space of

\tilde{J} -holomorphic maps $u : S^2 \rightarrow P$ in class σ_F and parametrized as sections of $P_\Lambda \rightarrow S^2$, and by $\mathcal{F} \subset \mathcal{M}_{0,2}^{\text{sec}}$ the subspace of constant sections.

Lemma 3.14. *Fix $\tilde{J} \in \mathcal{J}_S^n(P)$ and a fixed point component F . Then the evaluation map*

$$\text{ev} : \mathcal{M}_{0,2}^{\text{sec}}(P_\Lambda, \tilde{J}, \sigma_F) \rightarrow M_0 \times M_\infty$$

is transverse to $(\mathcal{N}_F \cap E^-) \times (\mathcal{N}_F \cap E^-) \subset M_0 \times M_\infty$ at all constant maps $u_x \in \mathcal{F}$. Moreover, if all the positive weights at F are $+1$, ev is a local diffeomorphism onto $(\mathcal{N}_F \cap E^+) \times (\mathcal{N}_F \cap E^+)$.

Proof. The definitions imply that \mathcal{F} has a neighborhood $\mathcal{N}(\mathcal{F})$ consisting of all holomorphic maps $\tilde{u} : S^2 \rightarrow \tilde{E}^+ \oplus \tilde{E}^-$ whose composite with the projection $\tilde{E}^+ \oplus \tilde{E}^- \rightarrow S^2 \times F$ is a holomorphic section of $\pi : S^2 \times F \rightarrow S^2$ in the class $[S^2 \times pt]$. Since $S^2 \times F \subset P_\Lambda$ has the product complex structure, \tilde{u} must project to some sphere $S^2 \times \{x\}$ for $x \in F$. Therefore \tilde{u} is a holomorphic section of the bundle $\tilde{E}^+ \oplus \tilde{E}^-|_{S^2 \times \{x\}}$. But this bundle is a sum of line bundles whose Chern classes are the nonzero weights of the S^1 action at F . The line bundles with negative Chern classes have no sections, but those with positive Chern classes have plenty. Moreover if all the positive weights are $+1$ then there is precisely one section of $\tilde{E}^+|_{S^2 \times \{x\}}$ through any pair of points lying in distinct fibers. The result follows. \square

Proof of Theorem 1.14. By Lemma 2.2, $u_\Lambda(\sigma_F) = -K(F)$ and $c_{\text{vert}}(\sigma_F) = m_F$. Therefore we may write

$$\mathcal{S}(\Lambda_K)(c^-) = \sum_{B \in H_2^S(M)} a_B \otimes q^{-m(F)-c_1(B)} t^{K(F)-\omega(B)},$$

where a_B is the contribution from the section $\sigma_F + B$. Let F' be a fixed component, and consider $c' \in H_*(F')$. Fix $B \in H_2^S(M)$ so that $\omega(B) \leq 0$. We want to show that $a_B = 0$ unless $B = 0$ and $F' = F$, in which case $a_B = c^+$. Since the classes $(c')^-$ form a basis for $H_*(M)$ it is enough to show that $a_B \cdot_M (c')^- = 0$ unless $B = 0$ and $F' = F$, in which case $a_B \cdot_M (c')^- = c \cdot_F c'$.

Choose a generic almost complex structure $J \in \mathcal{J}_S^n(M)$. By Lemma 4.5, the pair (K, g_J) is Morse regular, where g_J is the metric associated to J . We may assume without loss of generality that c and c' can be represented by generic submanifolds $C \subset F$ and $C' \subset F'$. By Lemma 4.6, the unstable manifolds $W_J^u(C)$ and $W_J^u(C')$ are pseudocycles. Moreover, by Proposition 4.8, $[W_J^u(C)] = c^-$, and $[W_J^u(C')] = (c')^-$. Let $\tilde{J} \in \mathcal{J}_S^n(P)$ be the associated almost complex structure on P_Λ .

Assume first that $B \neq 0$. By Lemma 3.12, the moduli space $\overline{\mathcal{M}}_{0,2}(P_\Lambda, \tilde{J}, \sigma_F + B; W_J^u(C), W_J^u(C'))$ contains no invariant chains. By Proposition 3.4 and Lemma 3.8, this implies that $a_B \cdot_M (c')^- = 0$.

Now suppose that $B = 0$ but $F \neq F'$. If $a_0 \neq 0$, then $\deg(a_0) = \deg(c^+) = \dim W_J^s(C)$. Since $\deg((c')^-) = \dim W_J^u(C')$, we see immediately that $a_0 \cdot_M (c')^- = 0$ for dimensional reasons unless $\dim W_J^u(C) + \dim W_J^s(C') = \dim M$. However, if $\dim W_J^u(C) + \dim W_J^s(C') = \dim M$, then by Lemma 3.12, the moduli space $\overline{\mathcal{M}}_{0,2}(P_\Lambda, \tilde{J}, \sigma_F; W_J^u(C), W_J^u(C'))$ contains no invariant chains. By Proposition 3.4 and Lemma 3.8, this implies that $a_B \cdot_M (c')^- = 0$.

Finally, assume that $B = 0$ and $F = F'$. By Lemma 3.12, the space

$$\overline{\mathcal{M}}_{0,2}(P_\Lambda, \tilde{J}, \sigma_F; W_J^u(C), W_J^u(C'))$$

contains no invariant chains except the constant sections σ_x for $x \in C \cap C'$. By Proposition 3.4 and Lemma 3.8, this implies that only these elements contribute to $a_0 \cdot_M (c')^-$. Now consider the full moduli space $\overline{\mathcal{M}}_{0,2}(P_\Lambda, \tilde{J}, \sigma_F)$. It follows from Lemma 3.14 that the evaluation map

$$\text{ev} : \overline{\mathcal{M}}_{0,2}(P_\Lambda, \tilde{J}, \sigma_F) \longrightarrow M_0 \times M_\infty$$

intersects $W_J^y(C) \times W_J^y(C')$ transversally in $c \cdot_F c'$ points. Hence $\text{GW}_{\sigma_F, 2}^P(c, c') = c \cdot_F c'$, as required.

3.4. The case of at most twofold isotropy. When the action is not semifree, the unstable manifolds given by an S^1 -invariant metric may not be pseudocycles. Therefore, we shall need to consider more general objects.

Definition 3.15. *Let C be a submanifold of a fixed component F with index α_F . A **downwards pseudocycle from C** is an S^1 -invariant weighted pseudocycle $f : Z_C^- \longrightarrow M$, or Z_C^- for short, of dimension $\dim C + \alpha_F$ such that $f(Z_C^-)$ lies in $M^{K(F)}$, $f(Z_C^-) \cap K^{-1}(K(F)) = C$, and $[Z_C^-] = [C]^-$. Here, $[C]^- \in H_*(M)$ is the downwards extension of $[C] \in H_*(F)$ constructed in Section 1.2.1.*

We show in Lemma 4.7 and Proposition 4.8 that given any generic submanifold $C \subset F$, there exists a downwards pseudocycle Z_C^- .

As in the semifree case, we investigate T^2 -invariant elements of the cut moduli space.

Lemma 3.16. *Consider a Hamiltonian circle action Λ_K on a compact symplectic manifold (M, ω) with at most twofold isotropy. Let \tilde{J} be a generic almost complex structure in $\mathcal{J}_S(P_\Lambda)$. Let F and F' be (not necessarily distinct) fixed point components, and let $C \subset F$ and $C' \subset F'$ be generic submanifolds. Fix a section class σ and consider the moduli space*

$$\overline{\mathcal{M}}_{0,2}(P_\Lambda, \tilde{J}, \sigma; Z_C^-, Z_{C'}^-).$$

If $u_\Lambda(\sigma) \leq -\frac{1}{2}(K(F) + K(F'))$, then the moduli space contains no invariant chains unless $\sigma = \frac{1}{2}(\sigma_F + \sigma_{F'})$ and F and F' lie in the same component of $M^{\mathbb{Z}/(2)}$.

Proof. Assume that there is an invariant chain from $x \in \overline{Z}_C$ to $y \in \overline{Z}_{C'}$ with root z in the class σ . We see immediately that

$$K(x) \leq K(F) \quad \text{and} \quad K(y) \leq K(F'),$$

with equality if and only if $x \in C \subset F$ and $y \in C' \subset F'$.

Let A' and A'' denote the classes represented by the invariant subchains from x to z , and from z to y , respectively. Since $A' + A'' + \sigma_z = \sigma$, by Lemma 2.2

$$\omega(A') + \omega(A'') - K(z) \leq -\frac{1}{2}(K(F) + K(F'))$$

Since the action has at most twofold isotropy, every bead of type (p, q) in the invariant chain from x to z has $q \leq 2$. Hence, by Lemma 3.9

$$\frac{1}{2}(K(z) - K(x)) \leq \omega(A'),$$

with equality impossible unless A' is the class of a chain of $(1, 2)$ beads from x to z and hence $A' = \frac{1}{2}(\sigma_z - \sigma_x)$. In particular, in this case x and z lie in the same

component of $M^{\mathbb{Z}/(2)}$. By similar reasoning,

$$\frac{1}{2}(K(z) - K(y)) \leq \omega(A''),$$

with equality impossible unless $A'' = \frac{1}{2}(\sigma_y - \sigma_z)$ and y and z lie in the same component of $M^{\mathbb{Z}/(2)}$.

Considering all five displayed inequalities together, it is clear that they must all be equalities. First, this means that $A' = \frac{1}{2}(\sigma_x - \sigma_z)$, $A'' = \frac{1}{2}(\sigma_y - \sigma_z)$, $\sigma_x = \sigma_F$, and $\sigma_y = \sigma_{F'}$; hence, $\sigma = \frac{1}{2}(\sigma_F + \sigma_{F'})$. Second, it implies that F and F' lie in the same component of $M^{\mathbb{Z}/(2)}$. \square

Proof of Theorem 1.16. Choose a generic almost complex structure $J \in \mathcal{J}_S(M)$. Let $\tilde{J} \in \mathcal{J}_S(P_\Lambda)$ be the associated almost complex structure on P_Λ . We may assume without loss of generality that c and c' can be represented by generic submanifolds $C \subset F$ and $C' \subset F'$, respectively. By Lemma 4.7 and Proposition 4.8 we can find downwards pseudocycles $Z_{\tilde{C}}$ and $Z_{\tilde{C}'}$ from C and C' as described above. Fix a section class σ so that $c_{\text{vert}}(\sigma) = u_\Lambda(\sigma) = 0$. If $K(F') \leq -K(F)$, then $0 = u_\Lambda(\sigma) \leq -\frac{1}{2}(K(F) + K(F'))$, so by Lemma 3.16, the moduli space $\overline{\mathcal{M}}_{0,2}(P_\Lambda, \tilde{J}, \sigma; Z_{\tilde{C}}, Z_{\tilde{C}'})$ contains no invariant chains unless $K(F) = -K(F')$, $m(F) = -m(F')$ and F and F' lie in the same component of $M^{\mathbb{Z}/(2)}$. The result now follows from Proposition 3.4 and Lemma 3.8. \square

The previous result concerns the term $a_{0,0} \otimes 1$ in $\mathcal{S}(\Lambda)(a)$ and hence gives information only in cases when we know that $a_{0,0} \neq 0$, for example if Λ is inessential. We next investigate the contribution from homologically visible components F for general Λ . Again our arguments work only if the isotropy at levels above F is at most twofold.

Lemma 3.17. *Consider a circle action Λ_K on a compact symplectic manifold (M, ω) with moment map $K : M \rightarrow \mathbb{R}$. Let \tilde{J} be a generic almost complex structure in $\mathcal{J}_S(P)$. Let F and F' be connected components of the fixed point set and let $C \subset F$ and $C' \subset F'$ be generic submanifolds. Assume $K(F') \leq K(F)$, that every positive weight at F is $+1$, and also that the isotropy for points w with $K(w) > K(F)$ is at most twofold. Consider the moduli space*

$$\overline{\mathcal{M}}_{0,2}(P_\Lambda, \tilde{J}, \sigma_F + B; Z_{\tilde{C}}, Z_{\tilde{C}'})$$

where $\omega(B) = 0$.

- If $B \neq 0$ or $F \neq F'$, there are no invariant chains in the moduli space.
- If $B = 0$ and $F = F'$, the only invariant chains are the constant sections u_x for $x \in C \cap C'$.

Proof. Assume that there is an invariant chain in class $\sigma_F + B$ from $x \in \overline{Z_{\tilde{C}}}$ to $y \in \overline{Z_{\tilde{C}'}}$ with root z . We see immediately that

$$K(x) \leq K(F) \quad \text{and} \quad K(y) \leq K(F'),$$

with equality if and only if $x \in C \subset F$ and $y \in C' \subset F'$.

Let A' and A'' denote the classes represented by the invariant subchains from x to z , and from z to y , respectively. Since $A' + A'' + \sigma_z = \sigma_F + B$, by Lemma 2.2

$$\omega(A') + \omega(A'') - K(z) + K(F) = \omega(B) = 0.$$

Since the isotropy for points w with $K(w) > K(F)$ is at most twofold, every (p, q) bead in the part of the invariant chain from x to z which lies at least partially above $K(F)$ has $q \leq 2$. Hence, by Lemma 3.9,

$$\frac{1}{2} \left(K(z) - \max\{K(F), K(x)\} \right) \leq \omega(A'),$$

with equality impossible unless $K(z) \geq K(F)$, $A' = \frac{1}{2}(\sigma_x - \sigma_z)$, and x and z lie in the same component of $M^{\mathbb{Z}/(2)}$. A similar reasoning applies to A'' . Hence,

$$\frac{1}{2} \left(K(z) - \max\{K(F), K(y)\} \right) \leq \omega(A''),$$

with equality impossible unless $K(z) \geq K(F)$, $A'' = \frac{1}{2}(\sigma_y - \sigma_z)$, and y and z lie in the same component of $M^{\mathbb{Z}/(2)}$.

Considering all five displayed equations together with the hypothesis $K(F') \leq K(F)$, it is clear that they must all be equalities.

This implies that $x \in F$ and $y \in F'$, that $K(z) \geq K(F)$, that $K(y) = K(F)$, and that x, y and z lie in the same connected component of $M^{\mathbb{Z}/(2)}$. Since all the positive weights at F are $+1$, this implies that in fact $z \in F$ and $y \in F$, so that $\sigma_x = \sigma_y = \sigma_z = \sigma_F$. Since $A' = \frac{1}{2}(\sigma_x - \sigma_z)$ and $A'' = \frac{1}{2}(\sigma_y - \sigma_z)$, this implies that $B = 0$, and the result follows. \square

Proof of Theorem 1.17. Let F' be a fixed component with $K(F') \leq K(F)$ and consider $c' \in H_*(F')$. Choose a generic almost complex structure $J \in \mathcal{J}_S^n(M)$. Let $\tilde{J} \in \mathcal{J}_S^n(P)$ be the associated almost complex structure on P_Λ . We may assume without loss of generality that c and c' can be represented by generic transversally intersecting submanifolds $C \subset F$ and $C' \subset F'$, respectively. By Lemma 4.7 and Proposition 4.8, we can find downward pseudocycles $Z_{\tilde{C}}$ and $Z_{\tilde{C}'}$ as in Definition 3.15. These are constructed to coincide with the unstable manifolds $W^u(C)$ and $W^u(C')$ near F . Since $J \in \mathcal{J}_S^n(M)$ is normalized near F , these unstable manifolds agree with neighborhoods of the zero section in the restrictions of $E^- \rightarrow F$ to C and C' respectively.

To show that $c_{0,0} \in F_{K(F)} H_*(M)$, it is enough to show that if $K(F') < K(F)$, then $c_{0,0} \cdot_M (c')^- = 0$ for all c' . By Proposition 3.4 and Lemma 3.8, it suffices to show that $\omega(B) = c_1(B) = 0$ then the moduli space $\overline{\mathcal{M}}_{0,2}(P_\Lambda, \tilde{J}, \sigma_F + B; Z_{\tilde{C}}, Z_{\tilde{C}'})$ contains no invariant chains. Since $F' \neq F$, this is immediate from Lemma 3.17.

Now suppose $F = F'$. Consider the moduli space $\overline{\mathcal{M}}^{cut} = \overline{\mathcal{M}}_{0,2}(P_\Lambda, \tilde{J}, \sigma_F + B; Z_{\tilde{C}}, Z_{\tilde{C}'})$. By Lemma 3.17, this is nonempty only if $B = 0$. Further, in this case the only invariant chains in $\overline{\mathcal{M}}^{cut}$ are the constant sections u_x for $x \in C \cap C'$. These sections form one of the connected components of $\overline{\mathcal{M}}^{cut}$, and by Proposition 3.4 we may ignore any other components. Thus we may suppose that $\overline{\mathcal{M}}^{cut}$ reduces to the compact manifold $C \cap C'$. If any of the negative weights along F are less than -1 the elements of $\overline{\mathcal{M}}^{cut}$ are not regular. As in the proof of Proposition 3.3, their cokernels fit together to form the obstruction bundle $\mathcal{E} \rightarrow \overline{\mathcal{M}}^{cut}$ of Equation (1). Because all positive weights along F are $+1$ and because the sets $\overline{Z_{\tilde{C}}}, \overline{Z_{\tilde{C}'}}$ coincide near F with the bundles $\overline{E^-|_C}, \overline{E^-|_{C'}}$, it follows from Lemma 3.14 that the full moduli space intersects $\overline{Z_{\tilde{C}}} \times \overline{Z_{\tilde{C}'}}$ transversally in $\overline{\mathcal{M}}^{cut}$. Hence standard theory implies that the regularized cut down moduli space represents the class $e(\mathcal{E}) \cdot_F [C \cap C'] =$

$(e(\mathcal{E}) \cap_F c) \cdot_F c'$ and that

$$\mathrm{GW}_{\sigma_F, 2}^{P_\Lambda}(c^-, (c')^-) = (e(\mathcal{E}) \cap_F c) \cdot_F c'.$$

The result follows. \square

4. PROOFS OF MAIN TECHNICAL LEMMAS

We now establish the main technical results used in the paper.

4.1. Invariant cycles in M . This section establishes the properties of the canonical extension classes c^\pm used in Theorems 1.14, 1.16, and 1.17. We first prove Lemma 1.12 which is used to construct canonical downwards and upwards extensions of the homology classes of the fixed point set, and then show how to define representing cycles for these classes that have the properties claimed in Definition 3.15.

4.1.1. Canonical classes. Let S^1 act on a symplectic manifold (M, ω) , with a moment map K which is proper and bounded below. Then K is Morse–Bott function with extraordinary properties. (For background information see [24].) First, K is equivariantly perfect, that is, the restriction map $H_{S^1}^*(M) \rightarrow H_{S^1}^*(M^{<\mu})$ is surjective for all $\mu \in \mathbb{R}$, where $M^{<\mu} := K^{-1}(-\infty, \mu)$. The same proof shows that the restriction to the fixed point set is injective. More specifically, given any $\tilde{Y} \in H_{S^1}(M)$, then $\tilde{Y}|_{M^{<\mu}} = 0$ if and only if $\tilde{Y}|_{F'} = 0$ for all fixed components F' with $K(F') < \mu$. The same argument also shows that $H_{S^1}^*(M)$ is equivariantly formal, that is, the restriction $H_{S^1}^*(M) \rightarrow H^*(M)$ is surjective.

Proof of Lemma 1.12. Let S^1 act on a compact symplectic manifold (M, ω) with moment map K . Let $F \subset M$ be any fixed component of index α ; and let $e_F^- \in H_{S^1}^\alpha(F)$ be the equivariant Euler class of the negative normal bundle to F . Given any cohomology class $Y \in H^i(F)$, we must show that there exists a unique cohomology class $\tilde{Y}^+ \in H_{S^1}^{i+\alpha}(M)$ so that

- (a): the restriction of \tilde{Y}^+ to $M^{<K(F)}$ vanishes,
- (b): $\tilde{Y}^+|_F = Y \cup e_F^-$, and
- (c): the degree of $\tilde{Y}^+|_{F'}$ in $H_{S^1}^*(pt)$ is less than the index $\alpha_{F'}$ of F' for all fixed components $F' \neq F$.

Moreover, we claim that these classes generate $H_{S^1}^*(M)$ as a $H_{S^1}(pt)$ module.

Since K is equivariantly perfect, we can find \tilde{Y}^+ satisfying (a) and (b). In fact, in general there will be many such \tilde{Y}^+ .

Enumerate the fixed sets other than F by F_1, \dots, F_k so that $K(F_j) \leq K(F_{j+1})$ for all j . Assume that \tilde{Y}^+ satisfies (c) for all F_j such that $j < i$. Let α_i denote the index of F_i , and let $m_-(F_i)$ denote the product of the negative weights at F_i . Then $e_{F_i}^-$, the equivariant Euler class of the negative normal bundle to F_i , is equal to $e_{F_i}^- = m_-(F_i) u^{\alpha_i/2} + \text{terms of degree } < \alpha_i$ in $H_{S^1}^*(pt)$, where $m_-(F_i) \neq 0$. Therefore, $\tilde{Y}^+|_{F_i} \in H_{S^1}^*(F_i)$ can be written uniquely as a sum $\tilde{X} + \tilde{X}'$, where \tilde{X} is a multiple of $e_{F_i}^-$ and the degree of \tilde{X}' in $H_{S^1}^*(pt)$ is less than α_i . Since K is equivariantly perfect, there exists $\tilde{Y}' \in H_{S^1}^*(M)$ so that $\tilde{Y}'|_{F_i} = \tilde{X}$ and $\tilde{Y}'_{F_j} = 0$ for all $j < i$. After subtracting \tilde{Y}' , we find a new \tilde{Y}^+ that satisfies (c) for all F_j such that $j \leq i$.

To see that \tilde{Y}^+ is unique, let \hat{Y} be the difference of two classes that satisfy (a), (b), and (c). Then the degree of $\hat{Y}|_{F'}$ in $H_{S^1}^*(pt)$ is less than the index of F' for *every* fixed component F' (and $\hat{Y}|_F = 0$). Let F_j be the smallest j such that $\hat{Y}|_{F_j} \neq 0$. Then, since the restriction to the fixed point set is injective, \hat{Y} vanishes when restricted to $H_{S^1}^*(M^{<K(F_j)})$. Hence, $\hat{Y}|_{F_j}$ is a multiple of $e_-(F_j)$. But this is impossible, so $\hat{Y}|_{F_i} = 0$ for all i . Hence, $\hat{Y} = 0$.

Finally, for any $Y \in H^*(F)$, since $\tilde{Y}^+|_{M^{<K(F)}} = 0$ the restriction of \tilde{Y}^+ to $M_{K(F)}$ is an element of $H_{S^1}^*(M^{K(F)}, M^{<K(F)})$. By injectivity, as Y ranges over $H^*(F)$, these classes generate $H_{S^1}^*(M^{K(F)}, M^{<K(F)})$ as a $H_{S^1}^*(pt)$ module. Hence, if we also let F vary over all fixed components, then they generate $H_{S^1}^*(M)$. \square

4.1.2. Morse cycles and equivariant cohomology. We shall work throughout with pseudocycles, and begin by recalling their definition from [15]. A **pseudocycle** of dimension d in a manifold M is a smooth map $f : V \rightarrow M$ from an oriented smooth d -dimensional manifold V to M whose Ω -limit set

$$V^\infty := \{x \in M : x = \lim_{i \rightarrow \infty} f(y_i), \text{ where } \{y_i\}_{i=1}^\infty \text{ has no limit point in } V\}$$

has codimension at least 2, i.e. it is in the image of a smooth map $g : W^{d-2} \rightarrow M$. Two pseudocycles $f_i : V_i \rightarrow M$ are bordant if they can be extended over a manifold W with boundary $V_1 \cup -V_2$ by a map whose Ω -limit set has dimension at most $d-1$. Any $f : V^d \rightarrow M$ is bordant to a map that intersects a given codimension d submanifold X of M transversally, i.e. $X \cap V^\infty = \emptyset$ and $f : V \rightarrow M$ meets X transversally. Moreover, because the boundary has codimension at least 2, each bordism class $[f, V]$ of pseudocycles defines a unique rational homology class $c(f, V)$. (In fact, it defines a unique integral class: see Schwarz [22].) We say that two such cycles of complementary dimension meet transversally if the closures of their images $f(V)$ and $f'(V')$ intersect only along their top strata $f(V)$ and $f'(V')$ and if these intersections are transverse in the usual way. It is shown in [15] that the intersection number $c(f, V) \cdot c(f', V')$ can be calculated by perturbing (f, V) to be transverse to (f', V') and then counting the points of intersection of f with f' in the usual way.

In this paper it is convenient to work with rational combinations of such pseudocycles. Therefore we make the following definition.

Definition 4.1. A **weighted pseudocycle** is a finite sum $\sum_i q_i f_i$, where $q_i \in \mathbb{Q}$ and $f_i : Z_i \rightarrow M$ is a pseudocycle as above. For short we sometimes forget the weights q_i and denote this pseudocycle by $f : Z \rightarrow M$, where $Z := \cup_i Z_i$ and $f|_{Z_i} := f_i$. We say that (f, Z) is S^1 -invariant iff the closure $\overline{f(Z)}$ of the image is invariant under the S^1 -action. We also sometimes omit the map f from the notation, denoting the cycle by Z and the closure of its image by \overline{Z} .

All the cycles considered in this paper (except for the virtual moduli cycle) are weighted pseudocycles.

Let K be a Morse-Bott function, and let g be any metric. Consider the **negative gradient flow**

$$\psi : \mathbb{R} \times M \rightarrow M$$

such that

$$\frac{\partial}{\partial t} \psi(t, x) = -\text{grad } f(\psi(t, x)) \quad \text{and} \quad \psi(0, x) = x \quad \text{for all } x \in M, t \in \mathbb{R}.$$

A **gradient trajectory** is a map $\gamma : \mathbb{R} \longrightarrow M$ such that

$$\frac{d}{dt}\gamma(t) = -\text{grad } f(\gamma(t)) \quad \text{for all } t \in \mathbb{R}.$$

More generally, a **broken gradient trajectory** is a set of gradient trajectories $\gamma_1, \dots, \gamma_n$ such that $\lim_{t \rightarrow \infty} \gamma_i = \lim_{t \rightarrow -\infty} \gamma_{i+1}$ for all i . (By convention, we allow the case $n = 1$.)

Given any critical component F , we define the **stable manifold** and the **unstable manifold**, respectively, by

$$W^s(F) = \{x \in M \mid \lim_{t \rightarrow \infty} \psi(t, x) \in F\}, \quad W^u(F) = \{x \in M \mid \lim_{t \rightarrow -\infty} \psi(t, x) \in F\}.$$

Define maps

$$\pi_+ : W^s(F) \longrightarrow F \quad \text{and} \quad \pi_- : W^u(F) \longrightarrow F$$

by

$$\pi_+(x) = \lim_{t \rightarrow \infty} \psi(t, x) \quad \text{and} \quad \pi_-(x) = \lim_{t \rightarrow -\infty} \psi(t, x).$$

Both π_+ and π_- are submersions; see [2].

Given a collection of distinct critical components F_1, \dots, F_k , we define a natural map

$$f : W^u(F_1) \times \dots \times W^u(F_{k-1}) \times W^s(F_2) \times \dots \times W^s(F_k) \longrightarrow (M^{k-1} \times F_2 \times \dots \times F_{k-1})^2$$

by

$$(14) \quad f(a_1, \dots, a_{k-1}, b_2, \dots, b_k) = (a_1, \dots, a_{k-1}, \pi_-(a_2), \dots, \pi_-(a_{k-1}), b_2, \dots, b_k, \pi_+(b_2), \dots, \pi_+(b_{k-1})).$$

Define $\mathcal{M}(F_1, \dots, F_k) = f^{-1}(\Delta)$. We can (and will) identify $\mathcal{M}(F_1, \dots, F_k)$ with tuples

$$(x_1, \dots, x_{k-1}) \subset M^{k-1}$$

such that $x_i \in W^s(F_i) \cap W^u(F_{i+1})$ and $\pi^+(x_i) = \pi^-(x_{i+1})$ for all i . More geometrically, $\mathcal{M}(F_1, \dots, F_k)$ consists of all tuples (x_1, \dots, x_{k-1}) for which there is a broken gradient trajectory $\gamma_1, \dots, \gamma_{k-1}$ from F_1 to F_k through F_2, F_3, \dots, F_{k-1} so that γ_i contains the point x_i for all i . Define maps

$$\pi_- : \mathcal{M}(F_1, \dots, F_k) \longrightarrow F_1 \quad \text{and} \quad \pi_+ : \mathcal{M}(F_1, \dots, F_k) \longrightarrow F_k$$

by

$$\pi_-(x_1, \dots, x_{k-1}) = \pi_-(x_1) \quad \text{and} \quad \pi_+(x_1, \dots, x_{k-1}) = \pi_+(x_{k-1}).$$

Definition 4.2. *We say that the pair (K, g) is **Morse regular** if f is transversal to the diagonal*

$$\Delta \subset (M^{k-1} \times F_2 \times \dots \times F_{k-1})^2$$

for every collection of critical components F_1, \dots, F_k .

In general, this is stronger than assuming that $W^s(F)$ and $W^u(F')$ intersect transversally for all critical sets F and F' , but it is equivalent if K is a Morse function.

If (K, g) is Morse regular, then by transversality, $\mathcal{M}(F_1, \dots, F_k)$ is a manifold of dimension $f_1 + \alpha_1 - \alpha_k$, where f_i is the dimension of F_i and α_i is the index of F_i . Note that the reparametrization group \mathbb{R}^{k-1} acts on the elements $(\gamma_1, \dots, \gamma_{k-1})$ in $\mathcal{M}(F_1, \dots, F_k)$ so that the set of points in M that lie on a broken trajectory in $\mathcal{M}(F_1, \dots, F_k)$ has dimension $\leq f_1 + \alpha_1 - \alpha_k - (k-1)$.

Lemma 4.3. *Let M be a compact manifold. Let $K : M \longrightarrow \mathbb{R}$ be a Morse-Bott function and let g be a metric so that the pair (K, g) is Morse regular. Let F and F' be distinct critical components. If $C \subset F$ and $C' \subset F'$ are generic submanifolds, there is no broken gradient trajectory from C' to C unless*

$$\dim W^s(C) + \dim W^u(C') > \dim M.$$

Proof. Assume that there is a broken trajectory from $F' = F_1$ to $F = F_k$ through critical components F_2, \dots, F_{k-1} . By genericity, we may assume that the maps $\pi_- : \mathcal{M}(F_1, \dots, F_k) \longrightarrow F_1$ and $\pi_+ : \mathcal{M}(F_1, \dots, F_k) \longrightarrow F_k$ are transverse to C' and C , respectively. Therefore, the set $X = C' \times_{\pi_-} \mathcal{M}(F_1, \dots, F_k) \times_{\pi_+} C$ is a manifold of dimension $c' + \alpha' + c - \alpha - f$, where c', c , and f denote the dimensions of C', C , and F , and α', α denote the index of F', F respectively. There is a proper effective action of \mathbb{R} on $\mathcal{M}(F_1, \dots, F_k)$ which moves x_1 along the gradient trajectory on which it lies; This induces an action on X . Hence, X is empty unless $c' + \alpha' + c - \alpha - f > 0$. Since $\dim W^u(C') = c' + \alpha'$, and $\dim W^s(C) = c - f - \alpha$, X is empty unless $\dim W^s(C) + \dim W^u(C') > \dim M$ as claimed. \square

We will also need the following lemma, which can be easily proved by a slight variation of the proof for the analogous fact in the Morse case.

Lemma 4.4. *Let M be a compact manifold. Let $K : M \longrightarrow \mathbb{R}$ be a Morse-Bott function and let g be a metric so that the pair (K, g) is Morse regular. Let C be a submanifold of a critical component F . Every point in $\overline{W^u(C)}$, the closure of the unstable manifold of C , lies on a broken trajectory beginning in C .*

Since we want the unstable manifold $W^u(C)$ to be S^1 -invariant, we next investigate gradient flows with respect to invariant metrics. In general, due to the presence of isotropy spheres, there may be no S^1 -invariant metric g so that the pair (K, g) is Morse regular, even if the moment map K is Morse. For example, consider the action $[z_0 : z_1 : z_2] \mapsto [e^{2\pi i t} z_0 : z_1 : e^{-2\pi i t} z_2]$ on \mathbb{CP}^2 and blow up the point $[0 : 1 : 0]$. The exceptional divisor Σ has isotropy group $\mathbb{Z}/(2)$ and contains two critical points, both of index 2. Any S^1 -invariant vector field must be tangent to Σ since if ϕ denotes the generator of the isotropy subgroup $d\phi$ acts as -1 in the directions normal to Σ . In particular, the gradient $\text{grad } K$ of K with respect to an invariant metric must be tangent to Σ and hence have trajectories joining two critical points of equal index. The following lemma shows that this is the only obstruction to finding a Morse regular pair (K, g_J) . Recall that $\mathcal{J}_S(M)$ is the space of smooth invariant ω -compatible almost complex structures on M that are normalized near the fixed point components F as described in Definition 2.3.

Lemma 4.5. *Let S^1 act semifreely on a compact symplectic manifold (M, ω) with moment map K . For a generic almost complex structure $J \in \mathcal{J}_S^n(M)$, the pair (K, g_J) is Morse regular, where g_J is the metric associated to J .*

Proof. Salamon and Zehnder show in [21, Theorem 8.1] that the gradient flow of any Morse function H on (M, ω) is Morse-Smale with respect to a generic metric of the form g_J , where J ranges over the set of all ω -compatible almost complex structures. We simply need to check that their argument continues to hold for Morse-Bott functions in the presence of a semifree S^1 -action.

Inspection of the proof of [21, Theorem 8.1] shows that the map f in equation (14) satisfies the required transversality condition provided that the tangent

space $T_J(\mathcal{J}_S)$ of the space $\mathcal{J}_S := \mathcal{J}_S(M)$ of allowable J is large enough. (See also Austin–Braam [2, Proposition B.2].) This tangent space $T_J(\mathcal{J}_S)$ is contained in the space of S^1 -invariant sections of the bundle End of anti- J -holomorphic endomorphisms of TM over M , and we need each gradient flow line γ to go through a point $x \in M$ such that there are elements $Y \in T_J(\mathcal{J}_S)$ whose value $Y(x)$ is an arbitrary element in End_x and whose support intersects γ in an arbitrarily small set. Since the isotropy group of x is trivial for all points on γ this is clearly the case; γ is transverse to the level sets of the moment map K and there are elements in $T_J(\mathcal{J}_S)$ with support in $K^{-1}(a, a + \varepsilon)$ for arbitrarily small ε and arbitrary value at x . If there were isotropy at x then this argument would fail because $Y(x)$ would have to be fixed by $d\phi$ for all ϕ in the isotropy group at x . \square

The following lemma is adapted from Schwarz [22].

Lemma 4.6. *Let S^1 act semifreely on a compact manifold M . Let $K : M \rightarrow \mathbb{R}$ be an S^1 -invariant Morse-Bott function and let g be an S^1 -invariant metric so that the pair (K, g) is Morse regular. Given a generic submanifold C of a fixed component F , the unstable manifold $W^u(C)$ is a pseudocycle.*

Proof. The unstable manifold $W^u(C)$ is a submanifold of dimension $c + \alpha$, where c is the dimension of C and α is the index of F . Hence, we must show that $\overline{W^u(C)} \setminus W^u(C)$ has dimension at most $c + \alpha - 2$.

Because C is generic, we may assume that C is transverse to the map $\pi^- : \mathcal{M}(F_1, \dots, F_n) \rightarrow F_1$ for every sequence of critical points $F = F_1, F_2, \dots, F_n$. Therefore, $X = C \times_{\pi^-} \mathcal{M}(F_1, \dots, F_n) \subset \mathcal{M}(F_1, \dots, F_n)$ is a manifold of dimension $c + \alpha - \alpha_n$, where α_n is the index of F_n .

There is a smooth proper action of \mathbb{R} on X , which moves the first coordinate x_1 along the gradient trajectory on which it lies. There is another smooth proper action of S^1 on X , which is given by the circle action on x_1 . If $n > 1$, the evaluation map $\text{ev} : X \rightarrow M$ defined by $\text{ev}(x_1, \dots, x_n) = x_n$ is constant along the orbits of these actions. Hence, the image of the evaluation map has dimension at most $c + \alpha - 2$.

By Lemma 4.4, every point in the closure $\overline{W^u(C)}$ lies on a broken trajectory that begins on C , that is, it lies in the image of the evaluation map for $X = C \times_{\pi^-} \mathcal{M}(F_1, \dots, F_n) \subset \mathcal{M}(F_1, \dots, F_n)$ for some sequence of fixed points $F = F_1, \dots, F_n$. Moreover, if the point does not lie $W^u(C)$ itself, then n must be greater than one. \square

Lemma 4.7. *Let S^1 act on a compact manifold M . Let $K : M \rightarrow \mathbb{R}$ be an S^1 -invariant Morse-Bott function. Given a generic submanifold C of a fixed component F of index α_F , there exists an S^1 -invariant weighted pseudocycle Z_C^- in $M^{K(F)}$ of dimension $\dim C + \alpha_F$ such that $Z_C^- \cap K^{-1}(K(F)) = C$.*

Proof. In this case, as illustrated by the example after Lemma 4.4 there may be no S^1 -invariant metric g so that the pair (K, g) is Morse regular. Instead, we begin with any S^1 -invariant metric g , and then consider an S^1 -invariant multivalued perturbation.

Briefly, the idea is this. Consider the space of all S^1 -invariant smooth multivalued vector fields Y on M . We will suppose for simplicity that Y is single valued everywhere except on a finite number of disjoint slices of the form $M^\mu \setminus M^{\mu-\varepsilon}$ that contain no critical points of K , and that at each point x in such a slice $Y(x)$ is a

finite set that is invariant under the action of the isotropy group at x . The smoothness condition means that the graph $\{(x, v); v \in Y(x)\}$ of Y is a union of smoothly embedded open subsets of Euclidean space. For example, in the case of the blow up of \mathbb{CP}^2 discussed at the beginning of this section, we allow Y to take two values $\pm v(x)$ when $x \in M^\mu \setminus M^{\mu-\varepsilon}$ is near the isotropy submanifold. It is easy to check that there are enough perturbations of this kind so that for generic small Y each solution $\gamma : \mathbb{R} \rightarrow M$ of the corresponding perturbed gradient flow relation

$$(15) \quad \frac{d}{ds}\gamma(s) \in \{-(\text{grad}_g K + Y)(\gamma(s))\}$$

is transverse to the level sets $K = \text{const}$ and regular in the sense of Salamon–Zehnder [21]. To keep the structure of the solution set as simple as possible we may assume that the number of elements in each set $Y(x)$ is constant and equal to N for all x lying in the interior of a slice, where N is the l.c.m. of the orders of the stabilizer subgroups of the S^1 -action. Then, a trajectory γ that goes from a point $\gamma(-\infty) \in F$ to F_{\min} passes through some number k of slices and hence satisfies one out of a set of N^k possible equations. Moreover, because the set of trajectories that do not reach F_{\min} lie in a closed subset of codimension at least 2, there is a neighborhood U of $\gamma(-\infty)$ in F such that the set of trajectories that start in U form a disjoint union of N^k submanifolds. Thus for each generic submanifold C in F the set $W_C^{u,Y}$ of solutions to (15) that start at C and end in F_{\min} is a manifold. As before, the transversality condition means that $W_C^{u,Y}$ intersects the corresponding stable manifolds transversally. (These are solutions to the relation $\frac{d}{ds}\gamma(s) \in \{(\text{grad}_g K + Y)(\gamma(s))\}$.) Hence the previous arguments apply to show that $W_C^{u,Y}$ is a pseudocycle. It is S^1 -invariant by construction. Therefore we define Z_C^- to be the weighted pseudocycle

$$Z_C^- := \frac{1}{N^k} W_C^{u,Y}.$$

This completes the construction. \square

Repeating the above construction for $-K$ we obtain upwards pseudocycles Z_C^+ . It remains to prove that these extensions represent the canonical extensions $[C]^\pm$.

Proposition 4.8. *Let S^1 act on a compact symplectic manifold (M, ω) with moment map $K : M \rightarrow \mathbb{R}$. Let C be a generic submanifold of a fixed component F . If the action is semifree, let g be an S^1 -invariant metric so that the pair (K, g) is Morse regular. Then $[W^s(C)] = [C]^+$. More generally, construct the weighted pseudocycle Z_C^+ as in Lemma 4.7. Then $[Z_C^+] = [C]^+$.*

Proof. Let Z^+ denote either the pseudocycle $W^s(C)$ or the weighted pseudocycle Z_C^+ , as appropriate. Let f be the dimension of F , i be the dimension of C , and let α be the index of F . Let $Y \in H^{f-i}(F)$ be the Poincaré dual to C . Let $Y^+ \in H^{f-i+\alpha}(M)$ denote the restriction to ordinary cohomology of the unique equivariant cohomology class $\tilde{Y}^+ \in H_{S^1}^{f-i+\alpha}(M)$ described in Lemma 1.12. Recall from Section 1.2.1 that the upwards extension $[C]^+$ is defined to be the Poincaré dual of the restriction of \tilde{Y}^+ to $H^{f-i+\alpha}(M)$.

Fix $N > d := f - i + \alpha$, and note that $\dim Z^+ = \dim M - d$. Since Z^+ is S^1 -invariant, it can be extended to a cycle $(Z^+)^N := S^{2N+1} \times_{S^1} Z^+$ in the finite

dimensional approximation $M_{S^1}^N := S^{2N+1} \times_{S^1} M$ to M_{S^1} . Denote by

$$\tilde{X}^N \in H^d(M_{S^1}^N)$$

the Poincaré dual of $(Z^+)^N$ in $M_{S^1}^N$. Clearly, the restriction of \tilde{X}^N to M is Poincaré dual to $[Z^+]$. Therefore, it is enough to show that \tilde{X}^N is the restriction of \tilde{Y}^+ to $M_{S^1}^N$. By the injectivity of the restriction maps

$$H_{S^1}^*(M) \longrightarrow H_{S^1}^*(M^{S^1}), \quad H_{S^1}^d(M) \longrightarrow H_{S^1}^d(M_{S^1}^N),$$

it is enough to show that the restriction $\tilde{X}^N|_F$ of \tilde{X}^N to $S^{2N+1} \times_{S^1} F$ satisfies the conditions (a), (b), and (c) of Lemma 1.12.

Because Z^+ has standard form near $C \subset F$, it is represented by the restriction over C of the positive normal bundle of F . Therefore $\tilde{X}^N|_F = Y \cup e_F^-$ as required by property (b) of Lemma 1.12. Clearly, $\tilde{X}^N|_{F'} = 0$ for all fixed components F' such that $K(F') < K(F)$. Therefore it suffices to check that \tilde{X}^N has property (c). Let F' be any fixed component other than F , and let α' be the index of F' . We wish to show that the degree of $\tilde{X}^N|_{F'}$ in $H^*((BS^1)^N) := H^*(\mathbb{CP}^N)$ is less than α' , or equivalently that the degree of $\tilde{X}^N|_{F'}$ in $H^*(F')$ is greater than $d - \alpha'$. To prove this, it is enough to show that if $X \subset F'$ is a generic submanifold of dimension $d - \alpha'$, then $(S^{2N+1}/S^1) \times X \subset S^{2N+1} \times_{S^1} F$ does not meet $S^{2N+1} \times_{S^1} \overline{Z^+}$. Hence it suffices to check that X does not meet $\overline{Z^+}$.

In the semifree case, Z^+ is the stable manifold $W^s(C)$ with respect to a generic metric g_J . By Lemma 4.4 every element in the closure $\overline{W^s(C)}$ lies on a broken geodesic ending at C . Therefore, by Lemma 4.3, $X \cap \overline{W^s(C)} \neq \emptyset$ only if $\dim X + \alpha' + \dim W^s(C) > \dim M$. Since by construction $\dim X + \alpha' + \dim W^s(C) = \dim M$, the intersection is empty.

In the general case, Z^+ is the sum of pseudocycles that are arbitrarily C^0 -close to $\overline{W^s(C)}$. Therefore, the argument above shows that it can be constructed so that its closure $\overline{Z^+}$ is disjoint from any finite set of manifolds $X_i \subset F'$ that span the homology group $H_{d-\alpha'}(F')$. The result follows. \square

4.2. Localization. In this section we show that when calculating Gromov–Witten invariants on a manifold with S^1 -action only the S^1 -invariant stable maps contribute. Here is a formal statement of our results.

Let (P, ω) be a closed symplectic manifold, and, given classes $a_1, \dots, a_k \in H_*(P)$, let $\alpha : Z \rightarrow P^k$ be a (possibly weighted) pseudocycle that represents their exterior product $a_1 \times \dots \times a_k \in H_*(P^k)$. Define

$$\overline{\mathcal{M}}_{0,k}(P, J, A; Z) := \text{ev}^{-1}(\overline{\alpha(Z)}),$$

where $\text{ev} : \overline{\mathcal{M}}(P, J, A) \rightarrow P^k$ is the evaluation map. First, we show that the calculation of the corresponding Gromov–Witten invariant can be localized in P in the following sense.

Lemma 4.9. *The invariant $GW_P(a_1, \dots, a_k; A)$ is a sum of contributions, one from each connected component of the cutdown moduli space $\overline{\mathcal{M}}_{0,k}(P, J, A; Z)$.*

Now consider the situation when (P, ω) carries an S^1 -action. We assume that J and the cycle α are S^1 -invariant (what this means for cycles is explained in Definition 4.1) so that the cutdown space $\overline{\mathcal{M}}_{0,k}(P, J, A; Z)$ also has an S^1 -action.

Proposition 4.10. *Let (P, ω) be a closed symplectic manifold with an S^1 -action $\{\phi_t\}_{t \in \mathbb{R}/\mathbb{Z}}$, and suppose that J and $\alpha : Z \rightarrow P^k$ are S^1 -invariant, where α represents $a_1 \times \cdots \times a_k$ as above. Then a connected component \mathcal{C} of $\overline{\mathcal{M}}_{0,k}(P, J, A; Z)$ makes no contribution to $GW_P(a_1, \dots, a_k; A)$ unless it contains an S^1 -invariant element.*

The next argument shows that this proposition is precisely what we need.

Proof of Proposition 3.4. Since the cycles Z, Z' are S^1 -invariant, the torus T^2 acts on the cutdown moduli space. Choose N greater than the order of any of the isotropy subgroups of the S^1 -action on M . Then the only sections of the bundle $P \rightarrow S^2$ that are invariant under the action of the subgroup $\{(Nt, t) : t \in S^1\}$ of T^2 are the constant sections σ_x at the fixed points $x \in M^{S^1}$. Therefore, it follows from Lemma 3.5 that the fixed points of this circle subgroup are the same as those for the action of the full torus. Hence by Lemma 4.9 and Proposition 4.10 the only components of the cutdown moduli space $\overline{\mathcal{M}}^{cut} := \overline{\mathcal{M}}_{0,k}(P, J, A; Z)$ that contribute to the GW invariant are those containing T^2 -invariant elements.

The second statement in Proposition 3.4 goes one step further, and claims that the GW invariant is a sum of contributions one from each component of the space of invariant elements $(\overline{\mathcal{M}}^{cut})^{T^2}$ in the cutdown moduli space. This is proved by applying the proof of Lemma 4.9 to the components of $(\overline{\mathcal{M}}^{cut})^{T^2}$. The details are straightforward, and are left to the reader. \square

We now explain the idea of the proof of Proposition 3.4 assuming for simplicity that $\overline{\mathcal{M}}_{0,k}(P, J, A; Z) =: \mathcal{C}$ is connected. If \mathcal{C} contains no S^1 -invariant elements, S^1 acts with finite stabilizers on \mathcal{C} and hence also on some neighborhood $\mathcal{N}(\mathcal{C})$ of \mathcal{C} in $\overline{\mathcal{M}} := \overline{\mathcal{M}}_{0,k}(P, J, A)$. We will see that we may give the quotient $\mathcal{N}(\mathcal{C})/S^1$ an orbifold structure and hence construct the regularized moduli cycle $\text{ev}^\nu : \overline{\mathcal{M}}^\nu \rightarrow P^k$ so that its subset

$$\mathcal{C}^\nu := (\text{ev}^\nu)^{-1}(\overline{\alpha(Z)})$$

has a neighborhood $\mathcal{N}(\mathcal{C}^\nu)$ that supports a *free* S^1 -action. Moreover, $\text{ev}^\nu : \mathcal{N}(\mathcal{C}^\nu) \rightarrow P^k$ is S^1 -equivariant. We will explain below the precise nature of the regularization $\overline{\mathcal{M}}^\nu$, but suppose for now that it is a closed manifold. It then suffices to apply the following fact. Suppose that a closed oriented manifold X supports a free S^1 -action and that $f : X \rightarrow P$ is equivariant. Then $f : X \rightarrow P$ may be perturbed to an equivariant map whose image is disjoint from the closure of the image of any invariant pseudocycle $\alpha : Z \rightarrow P$ of complementary dimension. This holds because locally X is the product of a transverse slice Y with S^1 , and it suffices to perturb the restriction $f|_Y$ so that it is disjoint from $\overline{\alpha(Z)}$ and then extend by equivariance. It is essential here that the action on X is free; otherwise one could not extend an arbitrary perturbation of $f|_Y$ to X .

Similar arguments have been used by many authors, for example in connection with the calculation of the Floer homology of a time independent small function: cf. Fukaya–Ono [5] and Liu–Tian [10]. The only difference is we are here dealing with an external S^1 -action (i.e. one on the range of the stable maps) rather than a reparametrization action which lives on the domain.

To carry out the details of the proof we will first describe how to construct the regularized (or virtual) moduli cycle $\text{ev}^\nu : \overline{\mathcal{M}}^\nu \rightarrow P^k$. We shall then prove Lemma 4.9, and finally the proposition. As in McDuff [12, 13], we will use the

regularization process of Liu–Tian [10]; readers can substitute their preferred constructions.

4.2.1. Branched pseudocycles. To start, we describe what kind of object the virtual moduli cycle is. For more details see [10, 12].

First, it is a d -dimensional partially smooth space $\iota_X : X_{sm} \rightarrow X$. Here X is a compact Hausdorff space (this is called the first topology), ι is a bijective continuous map, and X_{sm} is a union of a finite number of disjoint smooth manifolds X^i of dimensions $i \leq d$. The connected components of X_{sm} are called strata. Maps from one partially smooth space to another are given by commutative diagrams

$$\begin{array}{ccc} X_{sm} & \xrightarrow{\iota} & X \\ \downarrow & & \downarrow \\ Y_{sm} & \xrightarrow{\iota} & Y, \end{array}$$

but for short they are often written $f : X \rightarrow Y$. Also any compact smooth manifold is a partially smooth space in which P is given the usual topology and P_{sm} has one stratum. Hence a partially smooth map $f : X \rightarrow P$ is continuous when thought of as a map from the Hausdorff space X to the metric space P , and smooth when restricted to each stratum of X_{sm} . The virtual moduli cycle is a **compact branched partially smooth labelled pseudocycle**, or **(compact) branched pseudocycle** for short. This means it is a d -dimensional partially smooth space such that each d -dimensional stratum X_j is oriented, has a rational label, and fits together with $(d-1)$ -dimensional strata to form a branched manifold. More precisely, the closure $\overline{X_j}$ in $\iota(X^d \cup X^{d-1})$ of each component X_j of X^d can be given the structure of an oriented manifold with boundary; moreover, when one divides the top dimensional faces that meet each $(d-1)$ -dimensional component into two sets according to their orientations, the sum of the labels in each of these sets must be equal. If X has such structure then any map $f : X \rightarrow P$ represents a unique rational d -dimensional homology class. Just as with pseudocycles, there is an obvious notion of bordism.

A compact branched pseudocycle X is said to have a free S^1 -action with local slices if each point $x \in X$ has a neighborhood $U_{sm} \rightarrow U$ that is isomorphic to the product $Y_{sm} \times S^1 \rightarrow Y \times S^1$ with action $t \cdot (y, s) = (y, s + t)$.

The relevance of these definitions to the current problem is clear from the following lemma.

Lemma 4.11. *Suppose that the smooth manifold P supports an S^1 -action, that X is a compact branched labelled pseudocycle with free S^1 -action and that $f : X \rightarrow P$ is equivariant. Then f can be perturbed to an equivariant map that is disjoint from the closure of the image of any S^1 -invariant pseudocycle $\alpha : Z \rightarrow P$ of complementary dimension. Hence $f \cdot \alpha = 0$.*

Proof. As before, local equivariant perturbations of f may be constructed by perturbing f on the local slices Y . The perturbation is constructed by induction over the strata S , starting with those of lowest dimension. The perturbations have the form $f|_{Y \cap S} \mapsto \phi \circ f|_{Y \cap S}$ where ϕ is a suitable small diffeomorphism of P , and hence, even though we have little control over the way the strata in Y fit together, always extend from $Y \cap S$ to Y . Further details are left to the reader. \square

The above lemma does not extend to pseudocycles with free S^1 -action, since the definition of pseudocycle does not give us enough control of the boundary.

As an example, consider the standard S^1 -action on S^2 and take $f : \mathbb{C} \setminus \{0\} \longrightarrow S^2 = \mathbb{C} \cup \{\infty\}$ and $g : \{pt\} \longrightarrow \{0\}$. Then f is a pseudocycle representing the fundamental class, and it has a free S^1 -action in the sense that f is equivariant with respect to a free S^1 -action on its domain. Nevertheless $f \cdot g \neq 0$. The reason is the following. By definition, $f \cdot g$ is calculated by first perturbing f so that its boundary is *disjoint* from that of g and then counting transverse intersection points: see [15, Chapter 6]. In this example, the boundary of f meets $\text{im } g$ in an essential way and we cannot prevent this by a hypothesis concerning only the S^1 -action on the open set Z ; we must work with a closed domain.

4.2.2. Construction of the regularization. The basic idea in the construction of $\overline{\mathcal{M}}_{0,k}^\nu$ is to perturb the compactification $\overline{\mathcal{M}} := \overline{\mathcal{M}}_{0,k}(P, J, A)$ of $\mathcal{M}_{0,k}(P, J, A)$ to a cycle of the correct dimension. The analytic input to the construction explained below is the standard gluing result, see [10, 5] or [15, Chapter 10] for example; the rest of the construction is purely topological. The most important step in the proof of Proposition 3.4 is to construct the local uniformizers of Step 1 below so that they support a free S^1 -action.

The regularization process has four steps.

Step 1: Denote by $\tilde{\mathcal{B}}$ the space of k -pointed stable maps $\tilde{\tau} = (\Sigma(\mathbf{u}), \mathbf{u}, \mathbf{z})$ where $\tilde{\tau}$, though not necessarily J -holomorphic, has the property that the group of self-maps $\Gamma_\tau = \{\gamma : \mathbf{u} \circ \gamma = \mathbf{u}\}$ is finite. Let \mathcal{B} be the space of equivalence classes of such $\tilde{\tau}$. (More details are given in §4.2.3 below.) The elements of \mathcal{B} are organized into strata, depending on the topological types of their domains, and one can show that \mathcal{B} has the structure of an orbifold in the partially smooth category. (Objects in this category are spaces $B_{sm} \longrightarrow B$ with two topologies, where the first is Hausdorff and the second is a finite union of disjoint Banach manifolds.) Thus each point $\tau \in \mathcal{B}$ has a neighborhood U with a uniformizer (\tilde{U}, π, Γ) where $\pi : \tilde{U} \longrightarrow \tilde{U}/\Gamma = U$ identifies U with the quotient of \tilde{U} by the action of the finite group $\Gamma := \Gamma_\tau$. Since the elements in \tilde{U} are stable maps, constructing \tilde{U} amounts to choosing a consistent set of parametrizations for the elements $\tau \in U$. More details are given below. Because $\overline{\mathcal{M}} \subset \mathcal{B}$ is compact, it is contained in the union \mathcal{W} of a finite number U_1, \dots, U_N of such locally uniformized sets U , each of which is a neighborhood of some point $\tau \in \overline{\mathcal{M}}$. Throughout the construction one decreases the size of each U_i (and hence increases their number) as appropriate. Our notational convention is that objects living on the uniformizers \tilde{U} are designated with tildes.

Step 2: We interpret the operator $\overline{\partial}_J$ as a section of an orbibundle $\mathcal{L} \longrightarrow \mathcal{W}$. For each U there is a locally trivial bundle $\tilde{\mathcal{L}}_{\tilde{U}} \longrightarrow \tilde{U}$ on which the local isotropy group Γ acts. The fiber of $\tilde{\mathcal{L}}_{\tilde{U}}$ at $\tilde{\tau} = (\Sigma(\mathbf{u}), \mathbf{u}, \mathbf{z}) \in \tilde{U}$ is the space

$$\tilde{\mathcal{L}}_{\tilde{\tau}} := L^p(\Sigma(\mathbf{u}), \Lambda_J^{0,1} \otimes \mathbf{u}^*(TP))$$

of anti- J -holomorphic 1-forms on $\Sigma(\mathbf{u})$ of class L^p with values in $\mathbf{u}^*(TM)$. The perturbations used to define $\overline{\mathcal{M}}^\nu$ are built from sections of the local bundles $\tilde{\mathcal{L}}_{\tilde{U}} \longrightarrow \tilde{U}$. In order to extend these local sections, we construct another object $\tilde{\mathcal{L}} \longrightarrow \tilde{\mathcal{W}}$ from $\mathcal{L} \longrightarrow \mathcal{W}$ that is called a **multibundle**. Here $\tilde{\mathcal{L}} \longrightarrow \tilde{\mathcal{W}}$ is a collection of compatible maps $\tilde{\mathcal{L}}_I \longrightarrow \tilde{V}_I$, where I is a subset of the indexing set $\{1, \dots, N\}$ for the U_i , V_I is a suitable subset of $\cap_{i \in I} U_i$ and \tilde{V}_I (resp. $\tilde{\mathcal{L}}_I$) is the fiber product of the \tilde{U}_i (resp. $\tilde{\mathcal{L}}_i$) over V_I . The details of this construction are not important for

what follows. All we need to know is that each section $\tilde{s}(\nu)$ of $\tilde{\mathcal{L}} \longrightarrow \tilde{\mathcal{W}}$ (called a multisection) consists of a compatible collection $\{\tilde{s}(\nu)_I\}$ of multivalued sections of $\tilde{\mathcal{L}}_I \longrightarrow \tilde{V}_I$. It turns out that each $\tilde{s}(\nu)_I$ is single valued over the top strata, but may well be multivalued over lower dimensional strata.

Step 3: We construct a finite dimensional vector space R and a map $\nu \mapsto \tilde{s}(\nu)$ of R into the space of multisections of $\tilde{\mathcal{L}} \longrightarrow \tilde{\mathcal{W}}$ with the property that for generic small $\nu \in R$ the section $\bar{\partial}_J + \tilde{s}(\nu)$ is transverse to the zero section. The vector space R is a sum $\oplus_{i \in I} R_i$, where $\{U_i\}$ is an open covering of \mathcal{W} and for each i R_i is a suitable finite dimensional space of sections of $\tilde{\mathcal{L}}_{\tilde{U}_i} \longrightarrow \tilde{U}_i$. This space R_i is formed from the local obstruction bundle. The essential requirement is that for each stable map $\tilde{\tau} = (\Sigma(\mathbf{u}), \mathbf{u}, \mathbf{z}) \in \tilde{U}_i$ the subspace of $\tilde{\mathcal{L}}_{\tilde{\tau}}$ formed by the values $\{\nu(\tilde{\tau}) : \nu \in R_i\}$ projects onto the cokernel of the linearization $D_{\mathbf{u}}$ of $\bar{\partial}_J$ at \mathbf{u} . The fact that suitable finite dimensional spaces R_i exist is a consequence of the gluing construction and the compactness of $\overline{\mathcal{M}}$. To see this, choose for each $\tau \in U \subset \overline{\mathcal{M}}$:

- (a) a lift $\tilde{\tau} = (\Sigma(\mathbf{u}), \mathbf{u}, \mathbf{z}) \in \tilde{U}$ of τ ; and
- (b) a subspace $R_{\tilde{\tau}} \subset \tilde{\mathcal{L}}_{\tilde{\tau}}$ that covers the cokernel of $D_{\mathbf{u}}$.

Then extend the elements $\nu \in R_{\tilde{\tau}}$ by parallel translation along small paths in M to sections $\tilde{\tau}' \mapsto \nu(\tilde{\tau}')$ of $\tilde{\mathcal{L}}_{\tilde{U}}$ defined over some small neighborhood $\mathcal{N}(\tilde{\tau})$ of $\tilde{\tau}$ in \tilde{U} . By the gluing construction, the subspace

$$R_{\tilde{\tau}}(\tilde{\tau}') = \{\nu(\tilde{\tau}') : \nu \in R_{\tilde{\tau}}\} \subset \tilde{\mathcal{L}}_{\tilde{\tau}'}$$

projects onto $\text{coker } D_{\mathbf{u}'}$ when $\tilde{\tau}'$ is sufficiently close to $\tilde{\tau}$. Moreover, we can choose this subset $\tilde{U}_{\tilde{\tau}}$ of \tilde{U} to be invariant under the stabilizer group Γ_{τ} so that it has the form $\pi^{-1}(U_{\tau})$ for some neighborhood U_{τ} of τ in $\overline{\mathcal{M}}$. Therefore, by compactness of $\overline{\mathcal{M}}$, there is a finite set τ_i such that the corresponding pairs $(U_i, R_i) := (U_{\tau_i}, R_{\tilde{\tau}_i})$ have the required properties.

This defines the finite set of local pairs (U_i, R_i) , $1 \leq i \leq N$. One shows that each $\nu \in R_i$, when multiplied by a suitable cutoff function, gives rise to a multisection $\tilde{s}(\nu)$ of $\tilde{\mathcal{L}} \longrightarrow \tilde{\mathcal{W}}$. The most important point here is that the construction is local in $\overline{\mathcal{M}}$, i.e. for each $\nu \in R_i$ the section $\tilde{s}(\nu)_I = 0$ whenever the closure \bar{U}_i is disjoint from all the sets \bar{U}_j , $j \in I$. Now set $R := \oplus_i R_i$. It follows from the construction that the local multisections $\bar{\partial}_J + \tilde{s}(\nu)_I$ are transverse to the zero section for generic small $\nu \in R$. Hence the local zero sets $\tilde{Z}_I^{\nu} \subset \tilde{V}_I$ are submanifolds of the correct dimension d .

Step 4: We construct from the local zero sets \tilde{Z}_I^{ν} of $\bar{\partial}_J + \tilde{s}(\nu)$ a compact branched d dimensional pseudomanifold $\overline{\mathcal{M}}_{0,k}^{\nu}$. Its bordism class is independent of choices. There is a natural projection map

$$\text{proj} : \overline{\mathcal{M}}_{0,k}^{\nu}(P, J, A) \longrightarrow \mathcal{W}$$

such that each element in the image lies in the zero set of the multivalued section $\bar{\partial}_J + \nu$, and the evaluation map factors through this projection. Moreover, the strata in $\overline{\mathcal{M}}_{0,k}^{\nu}$ of dimensions $d, d-1$ project to the top stratum of \mathcal{W} , i.e. into stable maps whose domain has a single component. Therefore, when one evaluates the intersection number of $\text{ev} : \overline{\mathcal{M}}_{0,k}^{\nu}(P, J, A) \longrightarrow P^k$ with a cycle in P^k one will be counting rationally weighted curves $u : S^2 \longrightarrow P$ that satisfy a perturbed Cauchy–Riemann equation $\bar{\partial}_J u + \nu(u) = 0$.

Definition 4.12. *The Gromov–Witten invariant $GW_P(a_1, \dots, a_k; A)$ is the intersection number of the evaluation map $\text{ev} : \overline{\mathcal{M}}_{0,k}^\nu(P, J, A) \longrightarrow P^k$ with a generic representing pseudocycle $\alpha : Z \longrightarrow P^k$ for the class $a_1 \times \dots \times a_k$:*

$$GW_P(a_1, \dots, a_k; A) := \text{ev} \cdot \alpha.$$

It is zero by definition if the dimensional condition $\dim P + 2c_1(A) + 2k - 6 + \sum_i \dim a_i = k \dim P$ is not satisfied.

Lemma 4.9 claims that one would get the same answer by first cutting down the moduli space to $\overline{\mathcal{M}}(P, J, A; Z)$ and then regularizing each of its components separately.

Proof of Lemma 4.9. Let $\mathcal{C}_j, 1 \leq j \leq \ell$, be the connected components of $\overline{\mathcal{M}}_{0,k}(P, J, A; Z)$. By compactness there is $\varepsilon > 0$ so that the set

$$\mathcal{N}^{2\varepsilon} := \{\tau \in \overline{\mathcal{M}} : d(\text{ev}(\tau), \overline{\alpha(Z)}) \leq 2\varepsilon\},$$

where d is the metric in P^k , has ℓ connected components $\mathcal{N}_j^{2\varepsilon} \supset \mathcal{C}_j$. Now choose pairs (U_i, R_i) as in Step 3, where the open subsets $U_i \subset \mathcal{B}$ separate out the components \mathcal{C}_j in the following sense: if $\overline{U}_i \cap (\cup_j \mathcal{N}_j^\varepsilon \mathcal{C}_j) \neq \emptyset$ and $\overline{U}_k \cap (\overline{\mathcal{M}} \setminus (\cup_j \mathcal{N}_j^{2\varepsilon})) \neq \emptyset$ then U_i and U_k are disjoint. Then construct a regularization $\overline{\mathcal{M}}^\nu$ as described above. By the definition of $\mathcal{N}_j^\varepsilon$, the image under ev of the set $\overline{\mathcal{M}}^\nu \setminus \text{proj}^{-1}(\cup_j \mathcal{N}_j^\varepsilon)$ has distance at least ε from $\overline{\alpha(Z)}$ and so does not contribute to the intersection $\text{ev} \cdot \alpha$. On the other hand because the construction in Step 3 is local, the structure of $\mathcal{N}(\mathcal{C}_j^\nu) := \overline{\mathcal{M}}^\nu \cap \text{proj}^{-1}(\mathcal{N}_j^\varepsilon)$ depends only on the choices made for the open sets covering $\mathcal{N}_j^{2\varepsilon}$. Hence if we define the intersection number of $\text{ev} : \mathcal{N}(\mathcal{C}_j^\nu) \longrightarrow P^k$ with α as the local contribution of \mathcal{C}_j to the Gromov–Witten invariant $\text{ev} \cdot \alpha$, this invariant is the sum of local and independent contributions as claimed. \square

Corollary 4.13. *Let $\alpha : Z \longrightarrow P^k$ represent the class $a_1 \times \dots \times a_k$ in P^k . If $\overline{\mathcal{M}}_{0,k}(P, J, A; Z) = \emptyset$ then $GW_P(a_1, \dots, a_k; A) = 0$.*

4.2.3. The moduli space of stable maps as an orbifold. As preparation for the proof of Proposition 4.10, we describe the orbifold structure on the space of stable maps.

Let (T, E) be a finite tree where T denotes the set of vertices and the relation $E \subset T \times T$ describes the set of oriented edges. A genus zero stable map with k marked points modelled on T is a tuple

$$\left(\{u_\alpha\}_{\alpha \in T}, \{z_{\alpha\beta}\}_{\alpha E \beta}, \{z_i, \alpha_i\}_{1 \leq i \leq k} \right)$$

where $u_\alpha : S^2 \rightarrow P$ is a map, $z_{\alpha\beta} \in S^2$ denotes the point on the α -th sphere that attaches to the β -th sphere, and $z_i \in S^2$ is the i th marked point lying on the α_i -th sphere. Thus its domain $\Sigma(\mathbf{u})$ is the quotient of $S^2 \times T$ in which $(z_{\alpha\beta}, \alpha) \sim (z_{\beta\alpha}, \beta)$ whenever $\alpha E \beta$. We require that $u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha})$ whenever $\alpha E \beta$ so that the u_α induce a map $\mathbf{u} : \Sigma(\mathbf{u}) \rightarrow P$. The special points $\{z_{\alpha\beta} \mid \beta \in T\} \cup \{z_i \mid \alpha_i = \alpha\}$ on the α -th sphere are assumed distinct. The stability condition states that every ghost component (i.e. component of $\Sigma(\mathbf{u})$ on which \mathbf{u} is constant) has at least 3 special points.

Two such tuples $(u_\alpha, z_{\alpha\beta}, (z_i, \alpha_i))$ and $(u_{\alpha'}, z_{\alpha'\beta'}, (z'_i, \alpha'_i))$ modelled on T, T' are equivalent if there is a tree isomorphism $f : T \rightarrow T'$ and a collection $\phi_\alpha \in$

$\mathrm{PSL}(2, \mathbb{C})$, $\alpha \in T$, such that

$$u_{f(\alpha)} \circ \phi_\alpha = u_\alpha, \quad \phi_\alpha(z_{\alpha\beta}) = z_{f(\alpha)f(\beta)}, \quad (\phi_{\alpha_i}(z_i), f(\alpha_i)) = (z'_i, \alpha'_i).$$

We shall call such tuples (\mathbf{u}, \mathbf{z}) for short. The elements $\tau = [\mathbf{u}, \mathbf{z}]$ of the moduli space of stable maps \mathcal{B} are equivalence classes of such tuples. Each stratum consists of equivalence classes of stable maps modelled on a fixed tree T and has an obvious smooth topology. The Hausdorff topology on the whole space is discussed below.

We now describe Liu–Tian’s construction of the uniformizers $(\tilde{U}_\tau, \pi, \Gamma_\tau)$, where $\tau = [\mathbf{u}, \mathbf{z}]$ is modelled on T . Note that each uniformizer \tilde{U}_τ is a subset of the ambient space $\tilde{\mathcal{B}}$. Let us first suppose that $\Gamma_\tau = \{1\}$. Then the problem is to find a consistent way of parametrizing all the stable maps near τ . Choose a parametrization

$$\tilde{\tau} := (\mathbf{u}, \mathbf{z}) := (u_\alpha, z_{\alpha\beta}, (z_i, \alpha_i))$$

of τ . Add the minimum number of points $\mathbf{w} := (w_1, \alpha_{k+1}), \dots, (w_\ell, \alpha_{k+\ell})$ to the set of labelled points in $\tilde{\tau}$ to make its domain stable, i.e. so that each component has at least 3 special points. Pick out three of them for each component α and denote by Y the resulting subset of the $z_{\alpha\beta}, z_i$. The set \mathbf{w} is chosen to be invariant under the action of any element in Γ_τ that permutes the components of $\Sigma(\mathbf{u})$, but so that in each component no two are on the same orbit of the stabilizer of this component in Γ_τ . Next choose for each $j = 1, \dots, \ell$, a small open codimension-2 disc \mathbf{H}_j in P that is transverse to the image of \mathbf{u} at the points $\mathbf{u}(w_j)$. (This is possible because the ghost components are already stable and hence never contain any of the added points w_i .) If \mathcal{D}_T denotes the stratum in \mathcal{B} containing τ we define $\tilde{U}_\tau \cap \mathcal{D}_T$ to be a neighborhood of $\tilde{\tau}$ in the slice

$$\mathcal{S}_T := \left\{ (u'_\alpha, z'_{\alpha\beta}, (z'_i, \alpha_i)) \mid u'_{\alpha_{k+j}}(w_j) \in \mathbf{H}_j, z'_{\alpha\beta} = z_{\alpha\beta} \text{ if } z_{\alpha\beta} \in Y, z'_i = z_i \text{ if } z_i \in Y \right\}.$$

The domains of the stable maps $(\mathbf{u}', \mathbf{z}')$ near $\tilde{\tau}$ are formed from the domain of $\tilde{\tau}$ by gluing its components via the gluing parameters $a_{\alpha\beta} \in T_{z_{\alpha\beta}}(S_\alpha^2) \otimes T_{z_{\beta\alpha}}(S_\beta^2)$. (Here for convenience we denote the α th component by S_α^2 .) Assuming $r = |a_{\alpha\beta}|$ is sufficiently small we glue $S_\alpha^2 \setminus B_r(z_{\alpha\beta})$ to $S_\beta^2 \setminus B_r(z_{\beta\alpha})$ along their boundaries by a rotation determined by $\arg(a_{\alpha\beta})$. To describe this more precisely, let us suppose for simplicity that the tree T has two vertices $\{0, \infty\}$ and one edge, so that \mathcal{D}_T has codimension 2. We may suppose that

$$z_{0\infty} = \{0\} \in \mathbb{C} \cup \{\infty\} = S_0^2, \quad z_{\infty 0} = \{\infty\} \in \mathbb{C} \cup \{\infty\} = S_\infty^2.$$

There are two special points $y_{\alpha m} \in Y$ on each component that, together with $z_{0\infty}, z_{\infty 0}$, are fixed on the slice \mathcal{S}_T . By minimality the added points w_j (if there are any) form a subset of the four points $y_{01}, y_{02}, y_{\infty 1}, y_{\infty 2}$. Again, for the sake of clarity, let us suppose that there is one added point $w_1 := y_{01}$, and that $z_1 = y_{02}, z_2 := y_{\infty 1}, z_3 := y_{\infty 2}$. Consider the tuple $(a; u', (z'_i, \alpha_i)_{1 \leq i \leq k})$ where the domain S_a is the sphere

$$S_a := (S_0^2 \setminus B_r(0)) \cup (S_\infty^2 \setminus B_r(\infty)),$$

$u' : S_a \rightarrow P$ is close to \mathbf{u} in the obvious C^0 -sense and $z'_i \in S_{\alpha_i}^2 \setminus B_r \subset S_a$ is close to the image of z_i . Each such sphere S_a has a unique identification $\psi_a : S_a \rightarrow S^2$ with S^2 under which the four marked points w_1, z'_1, z'_2, z'_3 are taken to $0, 1, \infty, c(a)$, where $c(a)$ is their cross ratio. (Here we identify $w_1 \in S_0^2$ with its image in S_a^2 in the obvious way. Note also that one can define a so that $c(a) = a$.) Hence the tuple $(a; u', (z'_i, \alpha_i))$ can be written uniquely as a stable map $(u'', \mathbf{z}'') \in \mathcal{M}_{0,k}(S^2, A, J)$

where $u'' : S^2 \rightarrow P$ is the composite $u' \circ (\psi_a)^{-1}$, and $z_i'' = \psi_a(z_i')$, $i = 1, \dots, k$. Conversely, each stable map that is sufficiently close to $\tilde{\tau}$ does correspond to a unique tuple $(a; u', (z_i', \alpha_i))$ since the gluing parameter a is determined by the cross ratio of the four marked points $0, z_1'', z_2'', z_3''$. Therefore we may extend the slice \mathcal{S}_T by setting

$$\mathcal{S} := \left\{ (a; u', (z_i', \alpha_i)) \mid u' : S_a \rightarrow P, u'(w_{k+j}) \in \mathbf{H}_j, \right. \\ \left. z_i' = z_i \text{ if } z_i = y_{\alpha m} \text{ for some } \alpha, m \right\}.$$

Finally we define \tilde{U}_τ to be a neighborhood of $\tilde{\tau}$ in $\mathcal{S} \cup \mathcal{S}_T$. The projection to \mathcal{B} is given by dividing by the reparametrization group, i.e. by taking a stable map to its equivalence class. (We have not given a satisfactory description of the topology on \mathcal{B} : for this see [10, 12].)

Now suppose that $\Gamma_\tau \neq \{1\}$. We must extend the action of Γ_τ to \tilde{U}_τ . Suppose first that Γ_τ is a rotation group of order $n > 1$ with generator $\gamma \in \text{PSL}(2, \mathbb{C})$ that acts on a single component α_0 of $\Sigma(\mathbf{u})$. This component can have at most two special points y_i . Let us suppose that it has precisely two, say y_1, y_2 , and therefore one added point that we will call w_1 . We may suppose w_1 chosen so that the set $u_{\alpha_0}^{-1}(u_{\alpha_0}(w_1))$ contains n distinct points at which $du_{\alpha_0} \neq 0$. Choose disjoint little discs in $S_{\alpha_0}^2$ about these points that are permuted by γ . For any element $(\mathbf{u}', \mathbf{z}')$ that is close to $\tilde{\tau}$ and in the same stratum, $(u'_{\alpha_0})^{-1}(\mathbf{H}_1)$ is a collection of n points, one in each of the little discs. Therefore there is unique point w' in the little disc containing $\gamma(w_1)$ such that $\mathbf{u}'(w') \in \mathbf{H}_1$, and we define $\psi_{\mathbf{u}'}^\gamma \in \text{PSL}(2, \mathbb{C})^{|T|}$ to be the unique element that acts as the identity in all components except for the α_0 -th and there fixes y_1, y_2 and takes w_1 to w' . Then set

$$\gamma \cdot (\mathbf{u}', \mathbf{z}') = (\mathbf{u}' \circ \psi_{\mathbf{u}'}^\gamma, \mathbf{z}') \in \tilde{U}_\tau.$$

It is not hard to check that this does define an action of Γ_τ on a neighborhood of $\tilde{\tau}$ in $\tilde{U}_\tau \cap \mathcal{D}_T$.

It extends over the full neighborhood \tilde{U}_τ by acting on the gluing parameters a . We give a precise description in the case with $|T| = 2$ considered above. If $(a; u', w_1, (z_i', \alpha_i)) \in \mathcal{S}$ is sufficiently close to $\tilde{\tau}$, then $(u')^{-1}\mathbf{H}_1 \subset S_a$ consists of n points with precisely one, call it w' , in the little disc containing $\gamma(w_1)$. Because the map $a \mapsto c(a)$ is a diffeomorphism, there is a unique gluing parameter $\gamma(a)$ for which there is a biholomorphic map

$$\psi_\gamma : (S_{\gamma(a)}; y_{01} = w_1, y_{02}, y_{\infty 1}, y_{\infty 2}) \longrightarrow (S_a; y_{01} = w', y_{02}, y_{\infty 1}, y_{\infty 2}).$$

We define

$$\gamma \cdot (a; u', (z_i', \alpha_i)) := (\gamma(a); u' \circ \psi_\gamma, (\psi_\gamma^{-1}(z_i'), \alpha_i)) \in \mathcal{S}.$$

Alternatively, if we write the elements of \mathcal{S} in the form (u'', z_i'') where $u'' : S^2 \rightarrow P$ then

$$\gamma \cdot (u'', z_i'') = (u'' \circ h, h^{-1}(z_i'')), \quad h := \psi_a \circ \psi_\gamma \circ (\psi_{\gamma(a)})^{-1}.$$

Again, one can check that this gives a well defined action of Γ_τ on \tilde{U}_τ . Its continuity (which is somewhat tricky) is proved in [13, §4.2]; see also [12].

The construction for other groups Γ_τ is similar. We need to consider the case when Γ_τ acts in a single component with one or no special points; then consider

products of such actions; and finally consider an action that also permutes the components. These extensions are described in [10].

Proof of Proposition 4.10. Let \mathcal{C} be a component of $\overline{\mathcal{M}}_{0,k}(P, J, A; Z)$ on which the induced action of S^1 is locally free. We will show that its regularization $\mathcal{N}(\mathcal{C}^\nu)$ can be constructed so as to support a free S^1 -action. The result then follows from Lemma 4.11.

We show below that \mathcal{C} can be covered by S^1 -invariant sets W_j such that their uniformizers \widetilde{W}_j , as well as the uniformizers of all sets they meet, support a free S^1 -action with local slices \widetilde{Y}_j . Granted this, the construction of the (U_i, R_i) in Step 3 can be made so that the sections in R_i are S^1 -invariant. To see this, choose for each $\tau \in W_j$ with lift $\tilde{\tau}$ a suitable finite dimensional space $R_{\tilde{\tau}}$ of the fiber $\widetilde{\mathcal{L}}_{\tilde{\tau}}$, extend its elements to a neighborhood $\widetilde{Y}_{\tilde{\tau}}$ of $\tilde{\tau}$ in the slice \widetilde{Y}_j by parallel translation in P and then extend over the product $\widetilde{U}_{\tilde{\tau}} := \widetilde{Y}_{\tilde{\tau}} \times S^1$ using the S^1 -action on P . Since this action preserves J the transversality conditions continue to hold over the S^1 -orbit.

Hence the local zero sets \widetilde{Z}_I^ν all carry a free S^1 -action with local slices. The local virtual cycle $\mathcal{N}(\mathcal{C}^\nu)$ is made from these zero sets using partitions of unity, and one can check that its construction can be carried out in a way that respects the S^1 -action. Moreover the induced S^1 -action is free because each point in $\mathcal{N}(\mathcal{C}^\nu)$ projects to one of the sets \widetilde{V}_I and hence to $\widetilde{U}_i, i \in I$, where the action is free by construction: for details see Proposition 4.13 in [12].

Hence it remains to construct the W_j . To do this, we construct a different set of local uniformizers $(\widetilde{W}_\tau, \pi, \widetilde{\Gamma}_\tau)$ for a neighborhood $\mathcal{N}(\mathcal{C})$ of \mathcal{C} in \mathcal{B} whose stabilizer subgroups $\widetilde{\Gamma}_\tau$ incorporate not only the automorphism groups Γ_τ of the stable maps $\tau \in V_i$ but also the (finite) stabilizer subgroups $\text{Stab}(\tau) \subset S^1$ of the locally free S^1 -action on $\mathcal{N}(\mathcal{C})$. This amounts to defining an orbifold structure on the quotient $\mathcal{N}(\mathcal{C})_S := \mathcal{N}(\mathcal{C})/S^1$ whose elements are equivalence classes $[\Sigma(\mathbf{u}), \mathbf{u}, \mathbf{z}]_S$, where the equivalence relation \sim_S is generated by the previous relation \sim coming from the action of the reparametrization group together with the equivalence

$$(\Sigma(\mathbf{u}), \mathbf{u}, \mathbf{z}) \sim_S (\Sigma(\mathbf{u}), \phi_t \circ \mathbf{u}, \mathbf{z}), \quad t \in S^1,$$

where $\phi_t : P \rightarrow P$ denotes the action of $t \in S^1$.

The first task is to define the local group $\widetilde{\Gamma}_\tau$ at $\tau \in \mathcal{C}$. Choose a parametrization $\tilde{\tau} = (\mathbf{u}, \mathbf{z})$. Let $\Gamma_\tau := \{\gamma \in \text{Aut}(\Sigma(\mathbf{u})) : \mathbf{u} \circ \gamma = \mathbf{u}\}$ denote its automorphism group, and denote by N the order of the stabilizer subgroup $\text{Stab}(\tau)$ of τ in S^1 . Then define

$$\widetilde{\Gamma}_\tau := \{(\gamma, k) \in \text{Aut}(\Sigma(\mathbf{u})) \times \text{Stab}(\tau) \mid \mathbf{u} \circ \gamma = \phi_{k/N} \circ \mathbf{u}\}.$$

There are exact sequences

$$\text{Stab}(\tau)' \hookrightarrow \text{Stab}(\tau) \twoheadrightarrow \text{Stab}(\tau)'', \quad \Gamma_\tau \times \text{Stab}(\tau)' \hookrightarrow \widetilde{\Gamma}_\tau \twoheadrightarrow \text{Stab}(\tau)'',$$

where $\text{Stab}(\tau)' = \{t \in \mathbb{R}/\mathbb{Z} \mid \phi_t \circ \mathbf{u} = \mathbf{u}\}$ is the stabilizer subgroup of the image of \mathbf{u} in P .

We must show that every $\tau \in \mathcal{C}$ has a neighborhood W_τ with a uniformizer $(\widetilde{W}_\tau, \pi, \widetilde{\Gamma}_\tau)$ such that $(\widetilde{W}_\tau, \widetilde{\Gamma}_\tau)$ is equivariantly isomorphic to a product $\widetilde{Y}_\tau \times S^1$ with action induced by

$$(\gamma, k) \cdot (\mathbf{u}', t) := (\mathbf{u}' \circ \gamma, t - k/N).$$

Then the projection π given by $\pi(\mathbf{u}', \mathbf{z}', t) := \phi_t \circ (\mathbf{u}', \mathbf{z}')$ is well defined and S^1 -equivariant, and does quotient out by the action of $\tilde{\Gamma}_\tau$. For these formulas to make sense \tilde{Y}_τ must be invariant under the action of $\Gamma_\tau \times \text{Stab}(\tau)'$. To find such a slice \tilde{Y}_τ we will use the fact that, by hypothesis, $\text{Stab}(\tau)$ is finite.

Suppose first that $\Gamma_\tau = \{1\}$. Choose the added points w_j to be generic, i.e. so that $d\mathbf{u}(w_j) \neq 0$ and the stabilizers $\text{Stab}(\mathbf{u}(w_j))$ of the points $\mathbf{u}(w_j) \in P$ are as small as possible, and then choose the slices $\mathbf{H}_j \subset P$ to be $\text{Stab}(\mathbf{u}(w_j))$ -invariant. Note that $\text{Stab}(\tau)' \subseteq \text{Stab}(\mathbf{u}(w_j))$ for all j . Suppose in addition that it is possible to choose one of the added points, say w_2 , so that $\text{Stab}(\mathbf{u}(w_2))$ is finite. Then there is a $\text{Stab}(\mathbf{u}(w_2))$ -invariant codimension 1 disc X through $\mathbf{u}(w_2)$ that is transverse both to the S^1 -action and to \mathbf{H}_2 , and we set $\mathbf{H}'_2 := \mathbf{H}_2 \cap X$. Then

$$\tilde{Y}_\tau := \{\tilde{\tau}' \in \tilde{U}_\tau \mid \mathbf{u}'(w_2) \in \mathbf{H}'_2\}$$

is a slice for the induced local S^1 -action on \tilde{U}_τ ; in particular it is $\text{Stab}(\tau)'$ -invariant. Hence we may take

$$\tilde{W}_\tau := \tilde{Y}_\tau \times S^1, \quad W_\tau := \{\phi_t \cdot \tau' \mid \tau' \in \pi(\tilde{Y}_\tau), t \in S^1\}.$$

The projection $\tilde{W}_\tau \rightarrow W_\tau$ is given by $(\tilde{\tau}', t) \mapsto \phi_t \cdot \tau'$. Note that the uniformizer \tilde{W}_τ is no longer a subset of $\tilde{\mathcal{B}}$, but is defined so that it supports a free S^1 -action.

Suppose now that we cannot choose w_2 as above. (For example, there may be no need to add any w_j or the unstable components may all map into the fixed set.) Then, we choose any point $w_0 \in \Sigma(\mathbf{u})$ so that $\text{Stab}(\mathbf{u}(w_0))$ is finite. (This exists since $\text{Stab}(\tau)$ is finite.) We choose the slice X through $\mathbf{u}(w_0)$ as before and define $\tilde{Y}_\tau \subset \tilde{U}_\tau$ by the condition $\mathbf{u}'(w_0) \in X$. It is obvious what this means when $[\mathbf{u}', \mathbf{z}']$ is in the same stratum at τ . One extends to a neighboring strata as before. Note that in this case w_0 lies on a component with at least 3 special points.

Finally suppose that $\Gamma_\tau \neq \{1\}$. As before we treat the case when Γ_τ is cyclic and acting on one component of $\Sigma(\mathbf{u})$. Because this component contains at most 2 special points, w_0 (if it has been defined) always lies on some other component. Thus the only case that needs special consideration is when w_2 lies on the component on which Γ_τ acts and so equals the point previously called w_1 . But then we may simply repeat the previous construction for the action of Γ_τ , replacing \mathbf{H}_1 by $\mathbf{H}'_1 := \mathbf{H}_1 \cap X$. This defines an action of Γ_τ on \tilde{Y}_τ and hence completes the construction. \square

5. APPLICATIONS AND EXAMPLES

In the first section, we describe the small quantum cohomology of toric manifolds. Next, we work out $\mathcal{S}(\Lambda)$ in specific cases to illustrate what may happen when the hypotheses of the main theorems do not hold.

5.1. The small quantum homology of smooth toric varieties. This section describes the general form of a set of generators and relations for the small quantum cohomology ring $\text{QH}^*(M)$ of a toric manifold: see Proposition 5.2. In the case of a Fano variety the description is completely explicit; it is determined by a simple algorithm from the moment polytope Δ and agrees with Batyrev's presentation [3]. In the NEF case we show that $\text{QH}^*(M)$ is determined by a simple algorithm involving its moment polytope Δ together with the Seidel elements of the circle actions corresponding to the primitive outward normals η_1, \dots, η_N to the facets of Δ . (Of course, calculating the Seidel elements is a very nontrivial problem that we do not

attempt.) In the general case, the relations correspond to certain products of the Seidel elements but are not immediately determined by them. Our result elaborates on a very small part of Givental's work on the mirror conjecture: see Cox–Katz [4, Examples 8.1.2.2, 11.2.5.2]. Throughout we work with quantum cohomology with the Novikov ring coefficients defined in §2.2 though one can extend the result to the full Novikov ring: see Remark 5.5. Batyrev used complex coefficients; for a discussion of the relation of these coefficient systems see [4, 8.1.3].

Before beginning our computation, let us review a few facts about quantum cohomology. First, as in (8), define a valuation \check{v} on $\mathbb{Q}[x_1, \dots, x_N] \otimes \check{\Lambda}$ by

$$\check{v}\left(\sum_{d,\kappa} a_{d,\kappa} \otimes q^d t^\kappa\right) = \min\{\kappa \mid \exists d : a_{d,\kappa} \neq 0\}.$$

Lemma 5.1. *Let (M, ω) be a symplectic manifold. Fix $x_1, \dots, x_N \in H^*(M)$, and consider the natural homomorphisms of rings*

$$\begin{aligned} \theta : \mathbb{Q}[x_1, \dots, x_N] &\longrightarrow H^*(M), \quad \text{and} \\ \Theta : \mathbb{Q}[x_1, \dots, x_N] \otimes \check{\Lambda} &\longrightarrow \text{QH}^*(M). \end{aligned}$$

(i) *If θ is surjective, then Θ is also surjective. Further, given $z \in \text{QH}^*(M)$, there is $\tilde{z} \in \mathbb{Q}[x_1, \dots, x_N] \otimes \check{\Lambda}$ so that $\Theta(\tilde{z}) = z$ and so that $\check{v}(\tilde{z}) \geq \check{v}(z)$.*

(ii) *Let $p_1, \dots, p_m \in \mathbb{Q}[x_1, \dots, x_N]$ generate the kernel of θ , and suppose $q_1, \dots, q_m \in \mathbb{Q}[x_1, \dots, x_N] \otimes \check{\Lambda}$ are such that $\Theta(q_i) = 0$ and $\check{v}(p_i - q_i) > 0$ for all i . Then q_1, \dots, q_m generate the kernel of Θ .*

Proof. Fix $\hbar > 0$ such that \hbar is less than the energy $\omega(B)$ of every class $B \neq 0$ that contributes to the quantum multiplication, i.e. for which there is a nonzero three point Gromov–Witten invariant. Then $\check{v}(\alpha * \beta - \alpha \cup \beta) \geq \hbar$ for all $\alpha, \beta \in H^*(M)$, and hence

$$(16) \quad \check{v}(\Theta(\tilde{z}) - \theta(\tilde{z})) \geq \hbar, \quad \forall \tilde{z} \in \mathbb{Q}[x_1, \dots, x_N].$$

By possibly shrinking \hbar , we can also assume that $v^*(p_i - q_i) > \hbar$ for all $i = 1, \dots, m$.

Fix $z \in \text{QH}^*(M)$. To prove (i) it is enough to find $\tilde{z} \in \mathbb{Q}[x_1, \dots, x_N] \otimes \check{\Lambda}$ so that

$$\check{v}(z - \Theta(\tilde{z})) \geq \check{v}(z) + \hbar, \quad \text{and} \quad \check{v}(\tilde{z}) \geq \check{v}(z),$$

since then the argument can be completed by induction. Write

$$z = \sum_{i=1}^k z_i \otimes q^{d_i} t^{\kappa_i} + r,$$

where $\check{v}(r) \geq \check{v}(z) + \hbar$, $z_i \in H^*(M)$, $d_i \in \mathbb{Z}$, and $\kappa_i \geq \check{v}(z)$. Since θ is surjective, there exists $\tilde{z}_i \in \mathbb{Q}[x_1, \dots, x_N]$ so that $\theta(\tilde{z}_i) = z_i$. Then $\check{v}(z_i - \Theta(\tilde{z}_i)) \geq \hbar$ by (16); so let $\tilde{z} = \sum_{i=1}^k \tilde{z}_i \otimes q^{d_i} t^{\kappa_i}$.

Now fix $\tilde{y} \in \ker \Theta$. To prove (ii), it is enough to find $\tilde{z} \in \langle q_1, \dots, q_m \rangle$ so that

$$\check{v}(\tilde{z} - \tilde{y}) \geq \check{v}(\tilde{y}) + \hbar, \quad \text{and} \quad \check{v}(\tilde{z}) \geq \check{v}(\tilde{y}),$$

since then this argument can also be completed by induction. Write

$$\tilde{y} = \sum_{i=1}^k \tilde{y}_i \otimes q^{d_i} t^{\kappa_i} + \tilde{r},$$

where $\check{v}(\tilde{r}) \geq \check{v}(\tilde{y}) + \hbar$, $\tilde{y}_i \in \mathbb{Q}[x_1, \dots, x_N]$, $d_i \in \mathbb{Z}$, and $\check{v}(\tilde{y}) + \hbar > \kappa_i \geq \check{v}(\tilde{y})$. We may also assume that $(d_i, \kappa_i) \neq (d_j, \kappa_j)$ if $i \neq j$. Note that by (16)

$$0 = \Theta(\tilde{y}) = \sum \Theta(\tilde{y}_i \otimes q^{d_i} t^{\kappa_i}) + \Theta(\tilde{r}) = \sum \theta(\tilde{y}_i) \otimes q^{d_i} t^{\kappa_i} + \tilde{r}',$$

where $\check{v}(\tilde{r}') \geq \check{v}(\tilde{y}) + \hbar$. Therefore, for all i , $\theta(\tilde{y}_i) = 0$, and hence \tilde{y}_i lies in the ideal generated by p_1, \dots, p_m . Hence, there exists $\tilde{z}_i \in \langle q_1, \dots, q_m \rangle$, so that $\check{v}(\tilde{y}_i - \tilde{z}_i) \geq \hbar$. Let $\tilde{z} = \sum \tilde{z}_i \otimes q^{d_i} t^{\kappa_i}$. \square

We will now give a brief review of toric geometry. Good basic references are Cox–Katz [4, Ch 3] and Batyrev [3].

Consider a torus T with Lie algebra \mathfrak{t} and lattice ℓ . Let (M, ω) be a smooth toric variety with moment map $\Phi : M \rightarrow \mathfrak{t}^*$, chosen so that each of its components is mean normalized. Let $\Delta \subset \mathfrak{t}^*$ be the image of the moment map. Let D_1, \dots, D_N be the facets of Δ (the codimension one faces), and let $\eta_1, \dots, \eta_N \in \ell$ denote the outward primitive integral normal vectors.⁶ Let $\ell^* \subset \mathfrak{t}^*$ denote the lattice dual to ℓ . Let Σ be the set of subsets $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, N\}$ so that $D_{i_1} \cap \dots \cap D_{i_k} \neq \emptyset$. Define two ideals in $\mathbb{Q}[x_1, \dots, x_N]$:

$$P(\Delta) = \left\langle \sum (\xi, \eta_i) x_i \mid \xi \in \ell^* \right\rangle, \quad \text{and} \quad SR(\Delta) = \langle x_{i_1} \cdots x_{i_k} \mid \{i_1, \dots, i_k\} \notin \Sigma \rangle.$$

A subset $I \subseteq \{1, \dots, N\}$ is called **primitive** if I is not in Σ but every proper subset is. Clearly,

$$SR(\Delta) = \langle x_{i_1} \cdots x_{i_k} \mid \{i_1, \dots, i_k\} \subseteq \{1, \dots, N\} \text{ is primitive} \rangle.$$

The map which sends x_i to the Poincaré dual of $\Phi^{-1}(D_i)$ (which we shall also denote by $x_i \in H^2(M)$) induces an isomorphism

$$(17) \quad \mathbb{Q}[x_1, \dots, x_N] / (P(\Delta) + SR(\Delta)) \cong H^*(M, \mathbb{Q}).$$

Moreover, there is a natural isomorphism between $H_2(M; \mathbb{Z})$ and the set of tuples $(a_1, \dots, a_N) \in \mathbb{Z}^N$ such that $\sum a_i \eta_i = 0$, under which the pairing between such an element of $H_2(M; \mathbb{Z})$ and x_i is a_i . The linear functional η_i is constant on D_i ; let $\eta_i(D_i)$ denote its value. Under the isomorphism of (17) (extended to real coefficients)

$$(18) \quad [\omega] = \sum_i \eta_i(D_i) x_i, \quad \text{and} \quad c_1(M) = \sum_i x_i.$$

We are now ready to examine the quantum cohomology of a toric variety. The Seidel representation in cohomology is the homomorphism

$$S^* : \pi_1(\text{Ham}(M, \omega)) \rightarrow \text{QH}_{\text{ev}}(M; \Lambda)^\times, \quad \Lambda \mapsto \text{PD}(\mathcal{S}(\Lambda)),$$

where $\text{QH}_{\text{ev}}(M; \Lambda)^\times$ is the group of even units in $\text{QH}^*(M)$ and \mathcal{S} is the representation in homology. For each η_i define $\Phi^{\eta_i} : M \rightarrow \mathbb{R}$ to be the composite of the moment map $\Phi : M \rightarrow \mathfrak{t}^*$ with the linear functional $\eta_i \in \mathfrak{t} = \text{Hom}(\mathfrak{t}^*, \mathbb{R})$. Thus

⁶Choosing the $\eta_i \in \mathfrak{t}$ to be the outward rather than the inward normal is more natural in our context. For then the corresponding circle action has $\Phi^{-1}(D_i)$ as its maximal fixed point component, and it is this, rather than the minimal fixed point component, that is seen by the Seidel element: cf. Theorem 1.9. However, the authors of [3, 4] make the other choice, defining the polytope Δ by equations of the form $\{v \in \mathfrak{t}^* : \langle \eta_i, v \rangle \geq -a_i\}$. If we take the inward normals then in the definition of SR_Y in (20) β_I should be replaced by $-\beta_I$.

Φ^{η_i} is the moment map for the circle action Λ_i with tangent vector $\eta_i \in \mathfrak{t}$ and with $F_{\max} = \Phi^{-1}(D_i)$. Denote:

$$\mathcal{S}^*(\Lambda_i) = y_i \otimes q^{-1} t^{-\eta_i(D_i)} \in \mathrm{QH}_{\mathrm{ev}}(M; \Lambda)^\times.$$

By Theorem 1.9 and the formula given for Poincaré duality in §2.2, $y_i = x_i +$ higher order terms, where the terms are ordered by $\check{\nu}$.

Given any face of Δ , let $D_{j_1}, \dots, D_{j_\ell}$ be the facets that intersect to form this face. The **dual cone** is the set of elements in \mathfrak{t} which can be written as a positive linear combination of $\eta_{j_1}, \dots, \eta_{j_\ell}$. Every vector in \mathfrak{t} lies in the dual cone of a unique face of Δ . Therefore, given any subset $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, N\}$ there is a unique face of Δ so that $\eta_{i_1} + \dots + \eta_{i_k}$ lies in its dual cone. Let $D_{j_1}, \dots, D_{j_\ell}$ be the facets that intersect to form this unique face. Then there exist unique positive integers c_1, \dots, c_ℓ so that

$$\eta_{i_1} + \dots + \eta_{i_k} - c_1 \eta_{j_1} - \dots - c_\ell \eta_{j_\ell} = 0.$$

Batyrev showed that if I is primitive the sets I and $J = \{j_1, \dots, j_\ell\}$ are disjoint. Let $\beta_I \in H_2(M, \mathbb{Z})$ be the class corresponding to the above relation. By (18), we see that

$$\begin{aligned} c_1(\beta_I) &= k - c_1 - \dots - c_\ell, \quad \text{and} \\ \omega(\beta_I) &= \eta_{i_1}(D_{i_1}) + \dots + \eta_{i_k}(D_{i_k}) - c_1 \eta_{j_1}(D_{j_1}) - \dots - c_\ell \eta_{j_\ell}(D_{j_\ell}). \end{aligned}$$

Since $\eta_{i_1} + \dots + \eta_{i_k} = c_1 \eta_{j_1} + \dots + c_\ell \eta_{j_\ell}$, the corresponding circle actions are also equal. Using the fact that the Seidel representation is in fact a homomorphism, we have

$$\begin{aligned} y_{i_1} * \dots * y_{i_k} \otimes q^{-k} t^{-\eta_{i_1}(D_{i_1}) - \dots - \eta_{i_k}(D_{i_k})} = \\ y_{j_1}^{c_1} * \dots * y_{j_\ell}^{c_\ell} \otimes q^{-c_1 - \dots - c_\ell} t^{-c_1 \eta_{j_1}(D_{j_1}) - \dots - c_\ell \eta_{j_\ell}(D_{j_\ell})}. \end{aligned}$$

Therefore

$$y_{i_1} * \dots * y_{i_k} - y_{j_1}^{c_1} * \dots * y_{j_\ell}^{c_\ell} \otimes q^{c_1(\beta_I)} t^{\omega(\beta_I)} = 0.$$

Since x_1, \dots, x_N generate $H^*(M)$, by Lemma 5.1 the natural homomorphism

$$\Theta : \mathbb{Q}[x_1, \dots, x_N] \otimes \check{\Lambda} \longrightarrow \mathrm{QH}^*(M)$$

which takes x_i to the Poincaré dual of $\Phi^{-1}(D_i)$ is surjective. To compute $\mathrm{QH}^*(M)$, we need to find the kernel of Θ . By Lemma 5.1, there exists

$$(19) \quad Y_i = x_i + \text{higher order terms}$$

such that $\Theta(Y_i) = y_i$. Define an ideal $SR_Y(\Delta) \subset \mathbb{Q}[x_1, \dots, x_N] \otimes \check{\Lambda}$ by

$$(20) \quad SR_Y(\Delta) = \left\langle Y_{i_1} \dots Y_{i_l} - Y_{j_1}^{c_1} \dots Y_{j_k}^{c_k} \otimes q^{c_1(\beta_I)} t^{\omega(\beta_I)} \mid I = \{i_1, \dots, i_l\} \text{ is primitive} \right\rangle,$$

where the Y_i are as in (19). Note that $SR_Y(\Delta)$ depends on the Y_i . Additionally, even if the Seidel element y_i is known, it is not in general possible to describe its lift Y_i without prior knowledge of the ring structure on $\mathrm{QH}^*(M)$. On the other hand, SR_Y is clearly contained in the kernel of Θ . Moreover, Batyrev shows that $\omega(\beta_I) > 0$ for all primitive I . Hence, applying Lemma 5.1, we obtain the following proposition:

Proposition 5.2. *Let $\mathrm{QH}^*(M)$ denote the small quantum cohomology of the toric manifold (M, ω) . The map which sends x_i to the Poincaré dual of $\Phi^{-1}(D_i)$ induces an isomorphism*

$$\mathbb{Q}[x_1, \dots, x_N] \otimes \check{\Lambda} / (P(\Delta) + SR_Y(\Delta)) \cong \mathrm{QH}^*(M).$$

This is especially simple in the Fano case.

Example 5.3 (Fano toric varieties). Assume that M is Fano, i.e. that $c_1(B) > 0$ for every class $B \in H_2(M)$ with a holomorphic representative. In this case the higher order terms in $\mathcal{S}(\Lambda_i)$ vanish by part (iii) of Theorem 1.9. Therefore $y_i = x_i$ for all i , so that we may set $Y_i = x_i$. Hence

$$SR_Y(\Delta) = \langle x_{i_1} * \dots * x_{i_l} - x_{j_1}^{c_1} * \dots * x_{j_k}^{c_k} \otimes e^{\beta_I} \mid I = \{i_1, \dots, i_l\} \text{ is primitive} \rangle.$$

This gives exactly the formula for the small quantum cohomology of a Fano toric variety given by Batyrev and proved by Givental.

Example 5.4 (NEF toric varieties). Now assume that M is NEF, i.e. that $c_1(B) \geq 0$ for every class $B \in H_2(M)$ with a holomorphic representative. Now there may be higher order terms in the Seidel elements y_i . However, part (ii) of Theorem 1.9 implies that the higher order terms in $\mathcal{S}^*(\Lambda_i)$ have the form

$$\alpha_B \otimes q^{-1+c_1(B)} t^{\eta_i(D_i) + \omega(B)}$$

where $B \in H_2(M)$ satisfies $c_1(B) = 0$ or 1 . Since $\mathcal{S}^*(\Lambda_i)$ is homogeneous of degree 0, every nonzero α_B must have degree 0 or 2. Therefore α_B either lifts to the unit $\mathbb{1}$ in $\mathbb{Q}[x_1, \dots, x_N] \otimes \check{\Lambda}$ or to some linear combination of the x_i that is unique modulo the additive relations $P(\Delta)$. Hence the lifts Y_i of the Seidel elements y_i are determined by the linear relations $P(\Delta)$. The other information needed to determine the multiplicative structure of $\mathrm{QH}^*(M)$ is the set of primitive classes I . Thus, in the NEF case, once one knows the Seidel elements $\mathcal{S}(\Lambda_i)$, $i = 1, \dots, N$, there is an easy formula based on the combinatorics of its moment polytope Δ for the multiplicative relations in the quantum cohomology ring. This substitution of the Y_i for the x_i in the Stanley–Reisner ring SR_Y is one way of looking at Givental’s change of variable formulae as discussed in [4, 11.2.5.2]. \square

Remark 5.5. Often one wants to consider quantum cohomology with coefficients in a completion of the group ring of $H_2^S(M)$ rather than of a quotient of $H_2^S(M)$. Our methods give similar results in this case, but one must use a slightly different version of the Seidel representation. For more details see McDuff–Salamon [15, Chapter 11.4].

5.2. The Seidel representation: examples. The examples in this section show that even in the case of the simplest manifolds, namely rational ruled symplectic 4-manifolds, the Seidel element can be quite complicated. The first example is Fano. We show how lower order terms may appear in the formula for $\mathcal{S}(\Lambda_K)(a)$ and discuss a circle action with at most twofold isotropy. The second example illustrates the NEF case, in which, as already noted in Seidel [23], the expression for $\mathcal{S}(\Lambda)$ can have infinitely many nonzero terms. We also show what can happen when the isotropy has order greater than two.

Example 5.6 (The one point blowup of \mathbb{CP}^2). Fix $\mu \in (0, 1)$. Identify the one point blow up M_* of \mathbb{CP}^2 with the region

$$\left\{ (z_1, z_2) \in \mathbb{C}^2 \mid \frac{\mu^2}{\pi} \leq |z_1|^2 + |z_2|^2 \leq \frac{1}{\pi} \right\}$$

with boundaries collapsed along the Hopf flow, and give it the corresponding symplectic form ω_μ . Let $E \in H_2(M_*)$ denote the class of the exceptional divisor, let $L = [\mathbb{CP}^1]$, and let $B = L - E$ be the fiber class. Thus $\omega_\mu(L) = 1$. Let $p \in H_0(M)$ denote the homology class of a point, and let $\mathbb{1}$ be the generator of $H_4(M)$. The space M_* is a toric variety, where $T = S^1 \times S^1$ acts on M_* by $(\alpha_1, \alpha_2) \cdot (z_1, z_2) = (\alpha_1 z_1, \alpha_2 z_2)$. The standard complex structure J on M_* is T -invariant and is compatible with ω_μ . The moment map $\Phi : M \rightarrow \mathbb{R}^2$ is given by

$$\Phi(z_1, z_2) = (|z_1|^2 - \epsilon, |z_2|^2 - \epsilon), \quad \text{where } \epsilon = \frac{1 - \mu^6}{3(1 - \mu^4)}.$$

The primitive outward normals are

$$\eta_1 = (-1, 0), \quad \eta_2 = (0, -1), \quad \eta_3 = (1, 1), \quad \text{and } \eta_4 = (-1, -1).$$

Let Λ_i denote the circle action corresponding to η_i . Since the moment map for Λ_1 takes its maximum on the set $z_1 = 0$, Λ_1 is the action $(z_1, z_2) \mapsto (e^{-2\pi i t} z_1, z_2)$. Similar arguments give explicit formula for the other Λ_i . Since (M_*, J) is Fano, part (iii) of Theorem 1.9 implies that

$$\mathcal{S}(\Lambda_1) = \mathcal{S}(\Lambda_2) = B \otimes qt^\epsilon, \quad \mathcal{S}(\Lambda_3) = L \otimes qt^{1-2\epsilon}, \quad \text{and } \mathcal{S}(\Lambda_4) = E \otimes qt^{2\epsilon-\mu^2}.$$

There are two primitive subsets, namely $\{3, 4\}$ and $\{1, 2\}$. Since $\eta_3 + \eta_4 = 0$,

$$\mathbb{1} = \mathcal{S}(\Lambda_3) * \mathcal{S}(\Lambda_4) = L * E \otimes q^2 t^{1-\mu^2}.$$

Since $\eta_1 + \eta_2 = \eta_4$,

$$E \otimes qt^{2\epsilon-\mu^2} = \mathcal{S}(\Lambda_4) = \mathcal{S}(\Lambda_1) * \mathcal{S}(\Lambda_2) = B * B \otimes q^2 t^{2\epsilon}.$$

Therefore

$$(21) \quad B * B = E \otimes q^{-1} t^{-\mu^2} \quad \text{and} \quad L * E = \mathbb{1} \otimes q^{-2} t^{\mu^2-1}.$$

The circle action $(\Lambda_1)^{-1}$ also has a semifree maximum, namely the point $[(0, 1)] \in M_*$, the inverse image of the vertex $D_2 \cap D_3$. The holomorphic spheres C through F_{\max} all have $c_1(C) \geq 2$. Hence, again applying part (iii) of Theorem 1.9, we conclude

$$\mathcal{S}(\Lambda_1^{-1}) = p \otimes q^2 t^{1-\epsilon}.$$

Since $-\eta_1 = \eta_3 + \eta_2$,

$$p \otimes q^2 t^{1-\epsilon} = \mathcal{S}((\Lambda_1)^{-1}) = \mathcal{S}(\Lambda_3) \mathcal{S}(\Lambda_2) = B * L \otimes q^2 t^{1-\epsilon}.$$

Therefore,

$$(22) \quad B * L = p.$$

Note that equation (21) determines $\text{QH}_*(M_*)$ as a ring, but does not determine the product above. Together, equations (21) and (22) determine all possible products in $\text{QH}_*(M_*)$. In particular, using associativity, we find

$$p * p = L \otimes q^{-3} t^{-1}, \quad E * p = B \otimes q^{-2} t^{\mu^2-1}, \quad \text{and} \quad p * B = \mathbb{1} \otimes q^{-3} t^{-1}.$$

These products may also be derived directly from the 3-point Gromov–Witten invariants: it is not hard to check that the only nonzero invariants involving the classes p, B , and E are

$$\begin{aligned} \text{GW}_{L,3}(p, p, B) &= 1; & \text{GW}_{B,3}(p, E, E) &= 1; & \text{and} \\ \text{GW}_{E,3}(A_1, A_2, A_3) &= \pm 1 & \text{where } A_i &= E \text{ or } B. \end{aligned}$$

The natural action of $U(2)$ on \mathbb{C}^2 induces an action on M_* ; this action contains the torus T . Since $\pi_1(U(2)) = \mathbb{Z}$, this shows that, as elements of $\pi_1(\text{Symp}(M_*, \omega))$, $\Lambda_1 = \Lambda_2$. Hence

$$\Lambda_3 = (\Lambda_4)^{-1} = \Lambda_1^{-2} = \Lambda_2^{-2}$$

It is a worthwhile exercise to check that $\mathcal{S}(\Lambda_3) = \mathcal{S}(\Lambda_4)^{-1} = \mathcal{S}(\Lambda_1)^{-2} = \mathcal{S}(\Lambda_2)^{-2}$.

Since Λ_4 is semifree, we can also apply Theorem 1.14 to this action. It has $F_{\max} = E$, the exceptional divisor. Let $r \in H_0(E)$ be the homology class of a point. Then the downwards extension $r^- = B \in H_2(M)$, and the upwards extension $r^+ = p \in H_0(M)$. Then

$$\mathcal{S}(\Lambda_4)(r^-) = (E \otimes qt^{2\epsilon-\mu^2}) * B = r^+ \otimes qt^{2\epsilon-\mu^2} - E \otimes (qt^{2\epsilon-\mu^2})(q^{-1}t^{-\mu^2}).$$

This agrees with Theorem 1.14, but also shows that lower order terms can appear, even in this simple example. This lower order term comes from an invariant chain consisting of the sphere F_{\max} (in class E) together with a section σ_z for $z \in F_{\max}$.

Now consider the circle action Λ' corresponding to $\eta_1 + \eta_4 = (-2, -1)$. The corresponding moment map has a semifree maximum, namely the point $[(0, \mu)] \in M_*$ that maps down to $D_1 \cap D_4 \in \Phi(M)$. Hence part (i) of Theorem 1.9 applies, but part (iii) does not because there is a holomorphic sphere E through the maximum with $2c_1(E) = 2 \leq \text{codim } F_{\max} = 4$. Therefore our results do not rule out the presence of lower order terms in $\mathcal{S}(\Lambda')$ and indeed these exist: since $(-2, -1) = \eta_1 + \eta_4$,

$$\mathcal{S}(\Lambda') = \mathcal{S}(\Lambda_1) * \mathcal{S}(\Lambda_4) = B * E \otimes q^2 t^{3\epsilon-\mu^2} = p \otimes q^2 t^{3\epsilon-\mu^2} - E \otimes qt^{3\epsilon-2\mu^2}.$$

Observe also that Λ' has at most twofold isotropy, with isotropy submanifold $(M_*)^{\mathbb{Z}/(2)}$ equal to $\Phi^{-1}(D_2)$. One can check this by writing $(-2, -1) = 2\eta_4 - \eta_2$: as explained in the proof of Proposition 1.4 in §1.2 the coefficient of $-\eta_4$ in this expression equals the weight on the transverse edge D_2 . Therefore Theorem 1.17 applies to the fixed components $F_{13} := \Phi^{-1}(D_1 \cap D_3)$ and $F_{24} := \Phi^{-1}(D_2 \cap D_4)$, which are both isolated points. Since F_{13} is semifree the Euler class $e(F_{13})$ is nonzero. On the other hand, $e(F_{24}) = 0$. Further, if $c_{ij} \in H_0(F_{ij})$ denotes a generator, we find $(c_{13})^- = L$, while $(c_{24})^- = E$. Therefore Theorem 1.17 implies that $\mathcal{S}(\Lambda')(L)$ has a nontrivial summand $c_{0,0} \otimes t^{K'(F_{13})}$ where $c_{0,0} \cdot L = 1$ and K' denotes the moment map $\Phi^{\eta_1+\eta_4}$ of Λ' . (This is the contribution to $\mathcal{S}(\Lambda')(L)$ of the constant section at the homologically visible point F_{13} .) On the other hand, because F_{24} is not homologically visible, the constant section at F_{24} makes no contribution to $\mathcal{S}(\Lambda')(E)$ and so the coefficient of $qt^{K'(F_{24})}$ in $\mathcal{S}(\Lambda')(E)$ vanishes. This can be checked by direct calculation. For example $K'(F_{13}) = 3\epsilon - 1$ and $\mathcal{S}(\Lambda')(L) = \mathcal{S}(\Lambda')(E + B)$ contains one nonzero term of the form $a \otimes t^\kappa$, namely $B \otimes t^{3\epsilon-1}$.

The manifold M_* has many other toric structures; correspondingly there are many other elements of $\pi_1(\text{Ham}(M_*, \omega_\mu))$ that are represented by semifree circle actions. Indeed, whenever $\mu^2 > k/(k+1)$, there is an ω_μ -compatible complex structure J_k on M_* such that the underlying complex manifold (M_*, J_k) can be

identified with the projectivization $\mathbb{P}(L_k \oplus \mathbb{C})$, where L_k is the holomorphic line bundle over \mathbb{CP}^1 with Chern class $2k + 1$. The loop that rotates the fibers of \mathbb{C} by $e^{2\pi it}$ is semifree and represents the class $(4k + 2)\alpha$, where $\alpha = [-\Lambda_1] \in \pi_1(\text{Ham}(M_*, \omega_\mu))$.⁷ The classes $(2k + 1)\alpha$ are also represented by circle actions that preserve J_k and rotate the base of the ruled surface (M_*, J_k) . When $k = 0$, J_0 is the standard complex structure discussed above, and the representative for 2α is Λ_3 while the representative for α is Λ_1^{-1} . When $k > 0$ explicit formulas for these actions can be derived from the description of (M_*, J_k) as a toric manifold given in [1] §2.3. In this case these actions have 4 isolated fixed points, two each in the fibers lying above the fixed points of the base rotation. However, these fixed points are not semifree except when $k = 1$, in which case the fixed points in the fiber containing the overall minimum are semifree. In the next example we shall discuss a similar action on $S^2 \times S^2$ in detail.

Example 5.7 (Circle actions on $S^2 \times S^2$). Consider $M = \mathbb{CP}^1 \times \mathbb{CP}^1$ with the symplectic form $\omega_\mu = \mu\pi_1^*(\sigma) + \pi_2^*(\sigma)$, where π_i is projection onto the i 'th factor, σ is the standard symplectic form on \mathbb{CP}^1 with total area 1. Assume that $\mu \geq 1$. Define A and B in $H_2(M)$ by $A = [\mathbb{CP}^1 \times \{q\}]$ and $B = [\{q\} \times \mathbb{CP}^1]$, where $q \in \mathbb{CP}^1$. Note that $\omega_\mu(A) = \mu$ and $\omega_\mu(B) = 1$. Let $p \in H_0(M)$ denote the homology class of a point, and let $\mathbb{1}$ denote the generator of $H_4(M)$.

The standard action of the torus $T = S^1 \times S^1$ on (M, ω_μ) in which each S^1 -factor rotates the corresponding sphere has moment map $\Phi : M \rightarrow \mathbb{R}^2$ given by

$$\Phi([x_1 : x_2], [y_1 : y_2]) = \left(\mu \frac{|x_1|^2 - |x_2|^2}{|x_1|^2 + |x_2|^2}, \frac{|y_1|^2 - |y_2|^2}{|y_1|^2 + |y_2|^2} \right).$$

The primitive outward normals to the moment image $\Delta = \Phi(M)$ are

$$\eta_1 = (1, 0), \quad \eta_2 = (-1, 0), \quad \eta_3 = (0, 1), \quad \text{and} \quad \eta_4 = (0, -1).$$

Let Λ_i be the circle action associated to η_i .

Since the standard complex structure on $\mathbb{CP}^1 \times \mathbb{CP}^1$ is Fano and T -invariant, and since Λ_i acts semifreely for all i , by Theorem 1.9

$$\mathcal{S}(\Lambda_1) = \mathcal{S}(\Lambda_2) = B \otimes qt^{\frac{\mu}{2}}, \quad \text{and} \quad \mathcal{S}(\Lambda_3) = \mathcal{S}(\Lambda_4) = A \otimes qt^{\frac{1}{2}}.$$

Since $\eta_1 + \eta_2 = 0$ and $\eta_3 + \eta_4 = 0$, $\mathcal{S}(\Lambda_1) * \mathcal{S}(\Lambda_2) = \mathbb{1}$ and $\mathcal{S}(\Lambda_3) * \mathcal{S}(\Lambda_4) = \mathbb{1}$. This implies that

$$B * B = \mathbb{1} \otimes q^{-2}t^{-\mu} \quad \text{and} \quad A * A = \mathbb{1} \otimes q^{-2}t^{-1}.$$

Let $\Lambda' \subset S^1 \times S^1$ be the circle associated to $\eta_1 + \eta_3 := (1, 1)$. Then Λ' acts by the diagonal action and so is semifree. Since $c_1(C) \geq 2$ for every holomorphic sphere C ,

$$\mathcal{S}(\Lambda') = p \otimes q^2t^{\frac{1+\mu}{2}}.$$

Because $\mathcal{S}(\Lambda') = \mathcal{S}(\Lambda_1) * \mathcal{S}(\Lambda_3)$ we find $A * B = p$. As before, these products determine all the products in $\text{QH}_*(M)$. In particular

$$p * A = B \otimes q^{-2}t^{-1} \quad \text{and} \quad p * B = A \otimes q^{-2}t^{-\mu}.$$

We now describe a second toric structure on M . Let L_n denote the holomorphic line bundle over \mathbb{CP}^1 with Chern class n . Let M' be the projectivization of the bundle $L_2 \oplus \mathbb{C}$. Two commuting circles act naturally on M' . First, the standard circle action on \mathbb{CP}^1 lifts naturally to an action on $T^*(\mathbb{CP}^1) = L_2$, and hence to M' .

⁷The formula in [1] Lemma 2.11(i) is slightly incorrect.

Denote this circle action by Γ' . Another circle, say Γ'' , acts by rotating each fiber. The standard complex structure J_2 on M' is invariant under the resulting $S^1 \times S^1$ -action. Moreover, if we assume that $\mu > 1$, there exists a J_2 -compatible invariant symplectic form ω on M' so that M' is symplectomorphic to $(\mathbb{CP}^1 \times \mathbb{CP}^1, \omega_\mu)$, which we consider to be fibered over \mathbb{CP}^1 via projection to the first factor. In fact, we may assume that this symplectomorphism lifts the identity map on the base \mathbb{CP}^1 and is equivariant with respect to the action of Λ' on M and Γ' on M' : see for example [1]. Hence, we immediately conclude

$$\mathcal{S}(\Gamma') = \mathcal{S}(\Lambda') = p \otimes q^2 t^{\frac{1+\mu}{2}}.$$

Here, and elsewhere, we identify p , A , B , and $\mathbb{1}$ with their image in $H_*(M')$.

As described above, M' is a smooth toric variety with moment map Φ' . The moment image $\Delta' = \Phi'(M')$ is a quadrilateral with outward normals

$$\gamma_1 = (0, 1), \quad \gamma_2 = (0, -1), \quad \gamma_3 = (1, -1), \quad \text{and} \quad \gamma_4 = (-1, -1).$$

Further $(\Phi')^{-1}(D_1)$ is the diagonal in $M' \equiv \mathbb{CP}^1 \times \mathbb{CP}^1$ and so contains points that we will call v_{ss} and v_{nn} , where v_{ss} corresponds to $([0 : 1], [0 : 1])$, the pair (south pole, south pole), in $\mathbb{CP}^1 \times \mathbb{CP}^1$ and v_{nn} corresponds to (north pole, north pole). Similarly $(\Phi')^{-1}(D_2)$ is the antidiagonal and contains v_{sn}, v_{ns} . Indeed

$$\begin{aligned} (\Phi')^{-1}(D_1 \cap D_3) &= v_{nn}, & (\Phi')^{-1}(D_3 \cap D_2) &= v_{ns}, \\ (\Phi')^{-1}(D_2 \cap D_4) &= v_{sn}, & (\Phi')^{-1}(D_4 \cap D_1) &= v_{ss}. \end{aligned}$$

The moment image itself is

$$\Delta' = \{\alpha \in \mathbb{R}^2 \mid (\alpha, \gamma_i) \leq c_i\}, \quad \text{where}$$

$$c_1 = \frac{1}{2} + \frac{\mu}{2} - \epsilon, \quad c_2 = \epsilon + \frac{1}{2} - \frac{\mu}{2}, \quad c_3 = c_4 = \epsilon, \quad \text{and} \quad \epsilon = \frac{\mu}{2} + \frac{1}{6\mu}.$$

Let $\Gamma_i \subset S^1 \times S^1$ be the circle associated to γ_i . In our previous notation, $\Gamma' = \Gamma_1 + \Gamma_3$ and $\Gamma'' = \Gamma_1$.

The circle Γ_1 acts semifreely. Since every holomorphic sphere C which intersects F_{\max} has $c_1(C) \geq 2$, it follows from part (iii) of Theorem 1.9 that there are no lower order terms in $\mathcal{S}(\Gamma_1)$. Since $[F_{\max}] = [(\Phi')^{-1}(D_1)] = A + B$,

$$\mathcal{S}(\Gamma_1) = (A + B) \otimes q t^{\frac{1}{2} + \frac{\mu}{2} - \epsilon}.$$

The circle Γ_2 also acts semifreely. In this case, F_{\max} itself is a holomorphic sphere in class $A - B$, so $c_1(F_{\max}) = 0$. Therefore, part (iii) of Theorem 1.10 does not exclude lower order terms. On the other hand, every holomorphic sphere C with $c_1(C) \leq 1$ lies entirely in F_{\max} , so every term which contributes comes from a C which lies in F_{\max} . Indeed, since $\gamma_1 = -\gamma_2$,

$$\mathcal{S}(\Gamma_2) = \mathcal{S}(\Gamma_1)^{-1} = (A - B) \otimes \frac{q t^{\frac{1}{2} - \frac{\mu}{2} + \epsilon}}{1 - t^{1-\mu}} = (A - B) \otimes q t^{\frac{1}{2} - \frac{\mu}{2} + \epsilon} \left(1 + t^{1-\mu} + t^{2(1-\mu)} + \dots \right).$$

This calculation also appears in Remark 11.5 of [23].

Now consider Γ_3 . Once again, Theorem 1.10 does not rule out lower order terms. Since $\gamma_3 = \gamma_2 + (1, 0)$,

$$\mathcal{S}(\Gamma_3) = \mathcal{S}(\Gamma_2) * \mathcal{S}(\Gamma') = B \otimes q t^\epsilon - (A - B) \otimes q t^\epsilon \frac{t^{1-\mu}}{1 - t^{1-\mu}}.$$

A similar argument applies to Γ_4 .

Let's now pause for a moment to compare these results with the previous section. As above, let D_i denote the facet that corresponds to γ_i ; let x_i denote the Poincare dual of $\Phi^{-1}(D_i)$; note that $[(\Phi')^{-1}(D_3)] = [(\Phi')^{-1}(D_4)] = A$, $[(\Phi')^{-1}(D_1)] = A+B$, and $[(\Phi')^{-1}(D_2)] = A-B$. Converting the equations above into cohomology, and using this notation, we find:

$$\begin{aligned} \mathcal{S}^*(\Gamma_1) &= x_1 \otimes q^{-1}t^{-\frac{1}{2}-\frac{\mu}{2}+\epsilon}, \\ \mathcal{S}^*(\Gamma_2) &= x_2 \otimes \frac{q^{-1}t^{-\frac{1}{2}+\frac{\mu}{2}-\epsilon}}{1-t^{\mu-1}}, \\ \mathcal{S}^*(\Gamma_3) &= \left(x_3 - x_2 \otimes \frac{t^{\mu-1}}{1-t^{\mu-1}} \right) q^{-1}t^{-\epsilon}, \quad \text{and} \\ \mathcal{S}^*(\Gamma_4) &= \left(x_4 - x_2 \otimes \frac{t^{\mu-1}}{1-t^{\mu-1}} \right) q^{-1}t^{-\epsilon}. \end{aligned}$$

Thus in equation (19) we may take

$$Y_1 = x_1, \quad Y_2 = x_2 \otimes \frac{1}{1-t^{\mu-1}}, \quad \text{and} \quad Y_4 = Y_3 = x_3 - x_2 \otimes \frac{t^{\mu-1}}{1-t^{\mu-1}}.$$

We now look at $\mathcal{S}(\tilde{\Gamma})$ for the circle action $\tilde{\Gamma}$ associated with $\gamma = (1, 2)$, which has threefold isotropy. In notation introduced earlier, we can describe the fixed set of $\tilde{\Gamma}$ as consisting of the points v_{nn} (the maximum), the saddle points v_{ss}, v_{ns} and the minimum v_{sn} . The maximum is not semifree; in fact, because $(1, 2) = \eta_1 + 3\eta_3$, the diagonal $(\Phi')^{-1}D_1$ is stablized by $\mathbb{Z}/(3)$. Since the action does not have at most twofold isotropy, the arguments of Theorems 1.14 and 1.17 do not apply. We show that the conclusions of these theorems also fail. Since $(1, 2) = 2\gamma_1 + (1, 0)$,

$$\mathcal{S}(\tilde{\Gamma}) = \mathcal{S}(\Gamma_1)^2 * \mathcal{S}(\Gamma') = (p + p \otimes t^{1-\mu} + 2q^{-2}t^{-\mu}) \otimes q^2t^{\frac{1}{2}+\frac{3\mu}{2}-2\epsilon}.$$

First consider the minimum v_{sn} which is semifree. Then, in the notation of Theorem 1.17, $(v_{sn})^- = p$ and $(v_{sn})^+ = \mathbb{1}$ and so one might expect the leading order term of $\mathcal{S}(\tilde{\Gamma})(p)$ to come from the section σ_{sn} and so have the form $\mathbb{1} \otimes q^d t^\kappa$. But

$$\mathcal{S}(\tilde{\Gamma})(p) = (\mathbb{1} \otimes q^{-4}t^{-1-\mu} + \mathbb{1} \otimes q^{-4}t^{-2\mu} + 2p \otimes q^{-2}t^{-\mu}) \otimes q^2t^{\frac{1}{2}+\frac{3\mu}{2}-2\epsilon}$$

has the leading order term $p \otimes t^{\frac{1}{2}+\frac{\mu}{2}-2\epsilon}$. It is not hard to check that this term comes from the invariant chain

$$x = v_{sn} \xrightarrow{A-B} v_{ns} \xrightarrow{2B} v_{nn} \xrightarrow{\sigma_{nn}} y = v_{nn},$$

where σ_{nn} is the constant section at v_{nn} . This lies in class $A + B + [\sigma_{nn}] = A + B + [\sigma_{nn}] = [\sigma_{sn}] - B$ since $[\sigma_{sn}] - [\sigma_{nn}] = A + 2B$.

Next consider the semifree saddle point v_{ss} . Then $(v_{ss})^- = B$ and $(v_{ss})^+ = A + B$. Therefore, from Theorem 1.14 one would expect the leading order term in $\mathcal{S}(2\gamma + \tau_1 + \tau_2)(B)$ to be $(A + B) \otimes t^{K(v_{ss})}$, while in fact it is $(A + 2B) \otimes t^{K(v_{ss})}$. Since $(v_{ns})^+ = B$, one can get this extra term from an invariant chain going from $x \in (v_{ss})^-$ to $y \in (v_{ns})^-$ that lies in class $[\sigma_{ss}]$. Since $[\sigma_{nn}] = [\sigma_{ss}] - (A + B)$ such a chain is given by

$$x = v_{sn} \in (v_{ss})^- \xrightarrow{E} v_{ns} \xrightarrow{B} v_{nn} \xrightarrow{\sigma_{nn}} v_{nn} \xrightarrow{B} y = v_{ns} \in (v_{ns})^-.$$

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